Ising EoS from Alternative Expansion scheme (Ising-AltExS)

Micheal Kahangirwe

muses

UNIVERSITY OF HOUSTON

Collaborators: Claudia Ratti, Damien Price, Elena Bratkovskaya, Misha Stephanov,Pierre Moreau, Johannes Jahan Olga Soloveva, Steffen A. Bass

MUSES Collaboration Meeting 2023

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Context

- Available Equation of State with a critical point is limited in the range of Chemical potential $\mu_B \in (0\ ,\ 450\ \text{MeV})$
- An Equation of State with critical point that includes high chemical potential is needed for Hydro-Simulations

Objective

- To build an EoS with a Critical point from 3D-Ising model that captures large part of the phase diagram and matches lattice at low μ_B

Tools

- Alternative Expansion Scheme
- 3D -Ising Model

Physics Background

Lattice Results

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Taylor Expansion around $\mu_B = 0$

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$$\frac{P(T,\mu_B)}{T^4} = \sum_{n=0}^{\infty} \frac{1}{n!} \chi_n(T, \mu_B = 0) \left(\frac{\mu_B}{T} \right)^n$$

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$$c_n(T) = \frac{\chi_n^B(T, \mu_B = 0)}{n!} = \frac{1}{n!} \left(\frac{\partial}{\partial(\mu_B/T)} \right)^n (P/T^4) \Big|_{\mu_B=0}$$

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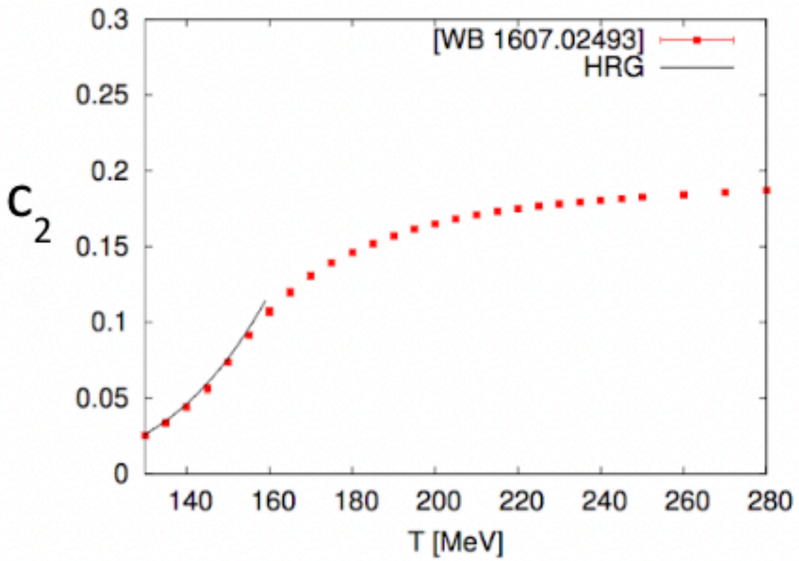

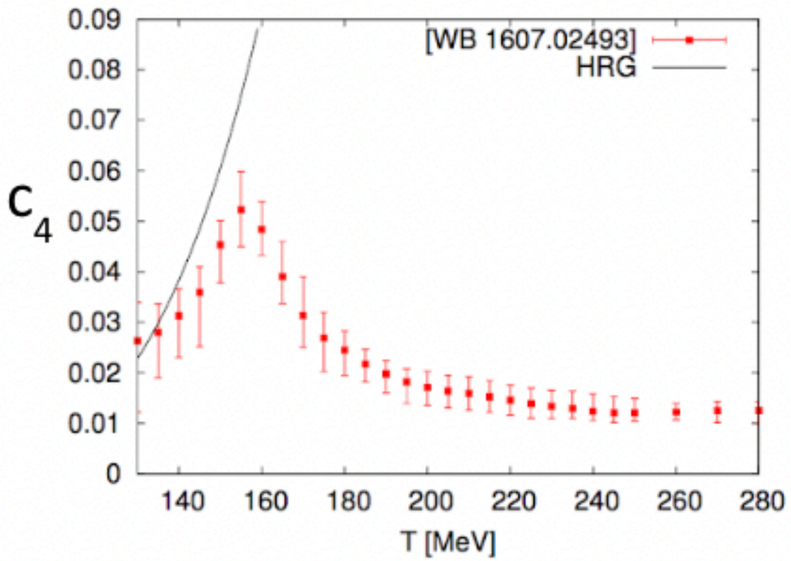

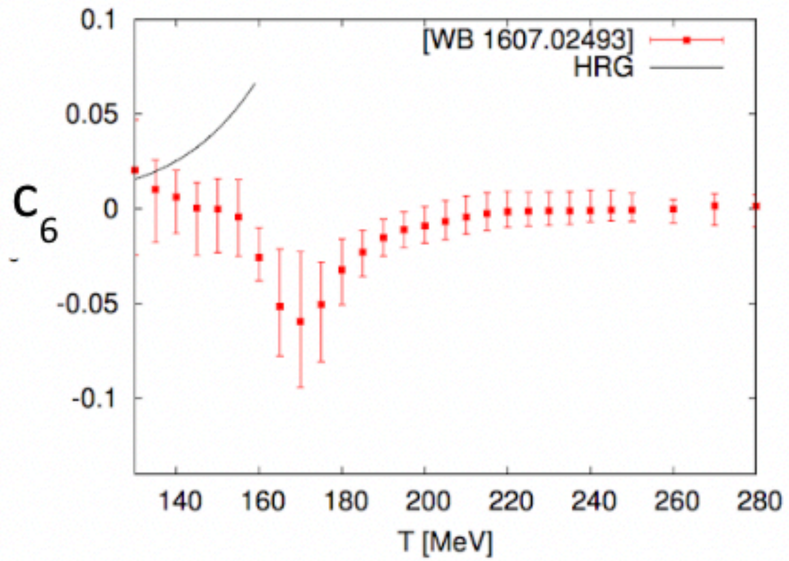

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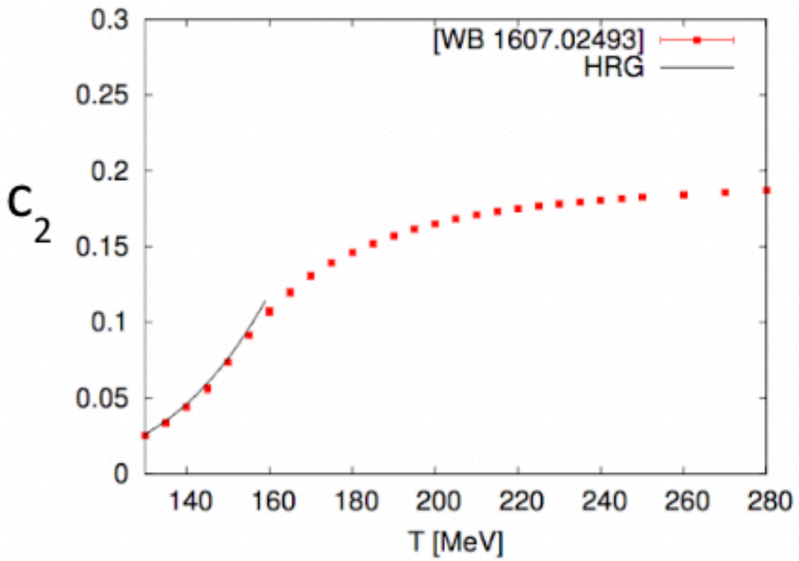
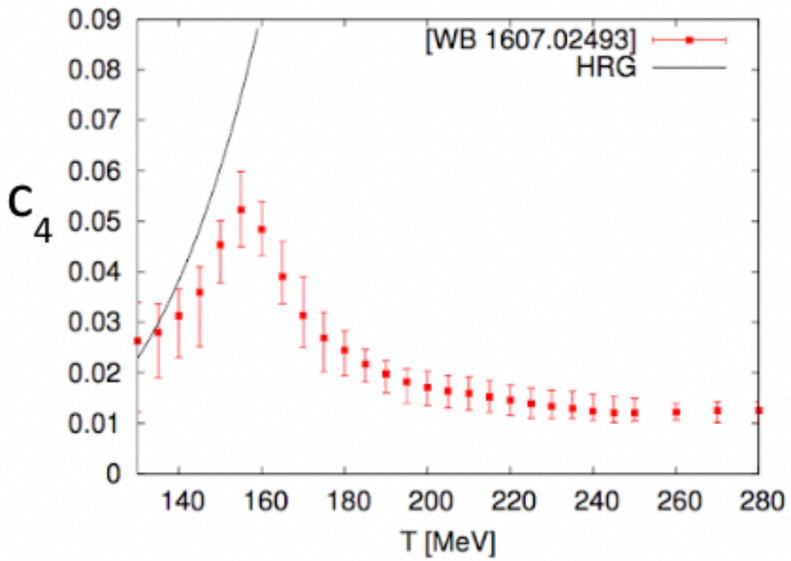
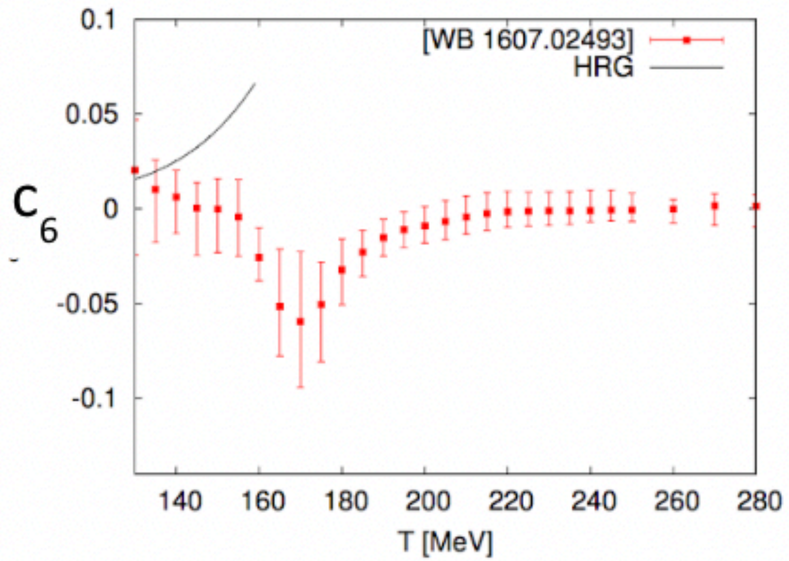
Limitations







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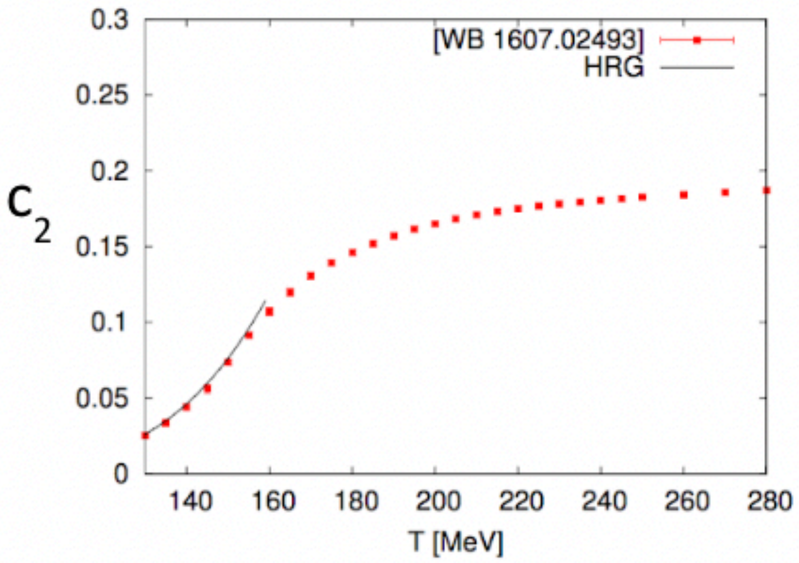
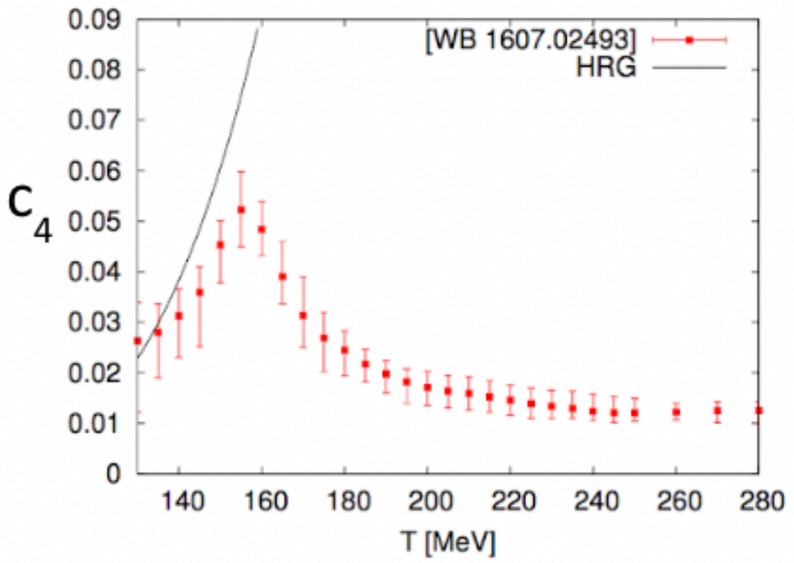
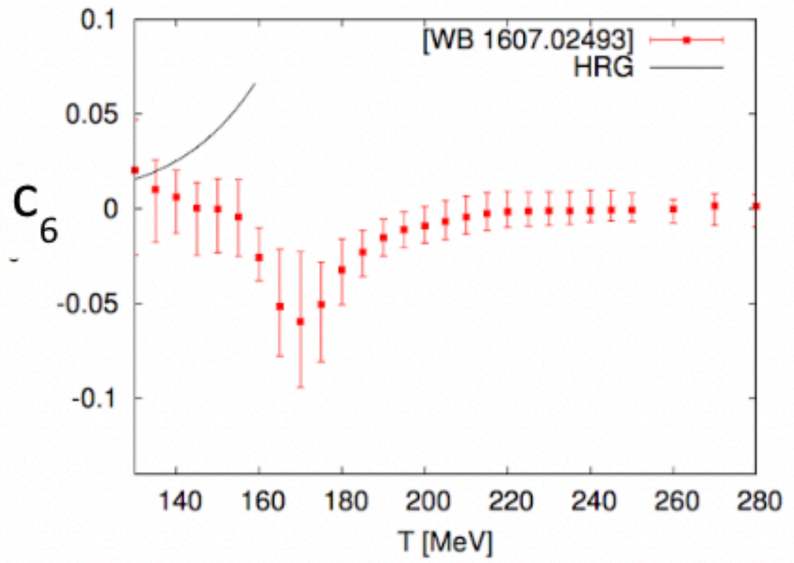
Limitations

- **Currently limited to $\frac{\mu_B}{T} \leq 2.5$ despite great Computational Power**
- **Adding one more Higher-Order term does not help in convergence**
- **Taylor expansion is carried out at T= constant and doesn't cope well with μ_B-dependent transition temperature**







Alternative Expansion scheme

Simulating at Imaginary μ_B

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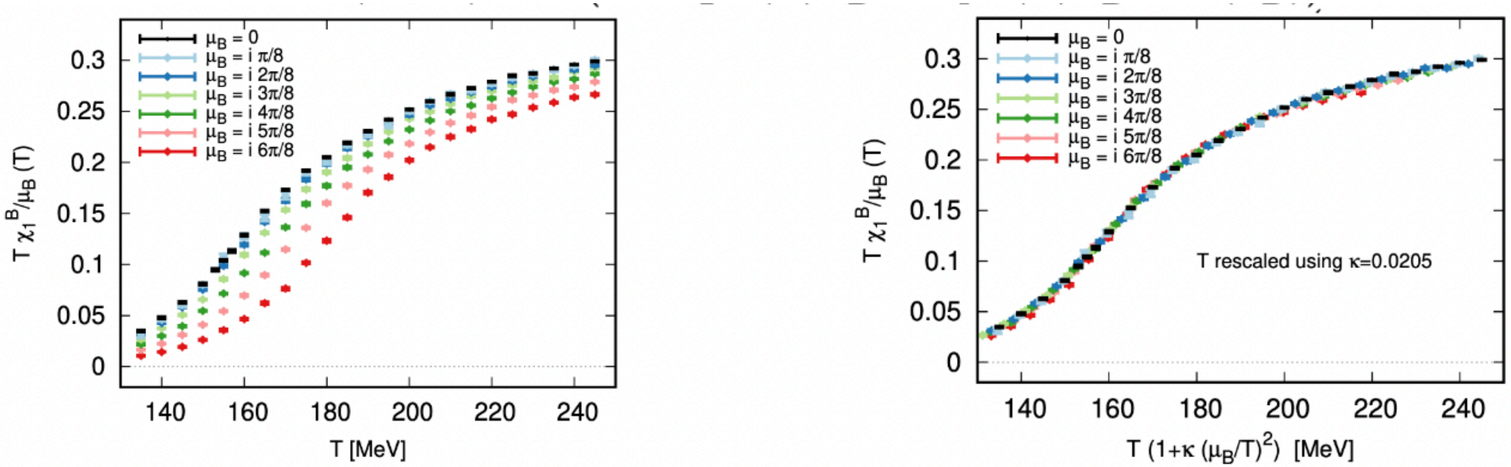

[Borsányi, S. et al. PRL (2021)]

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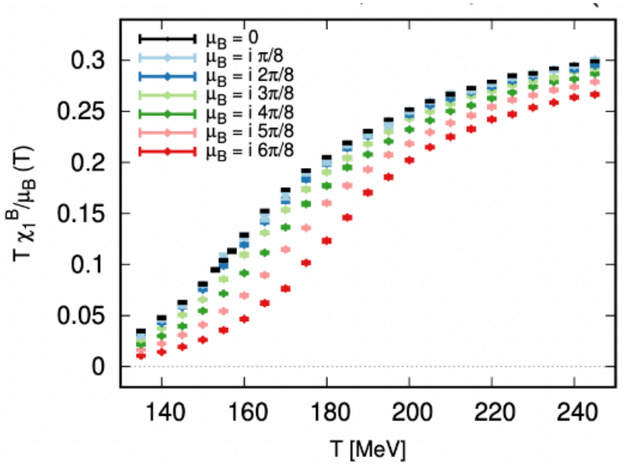

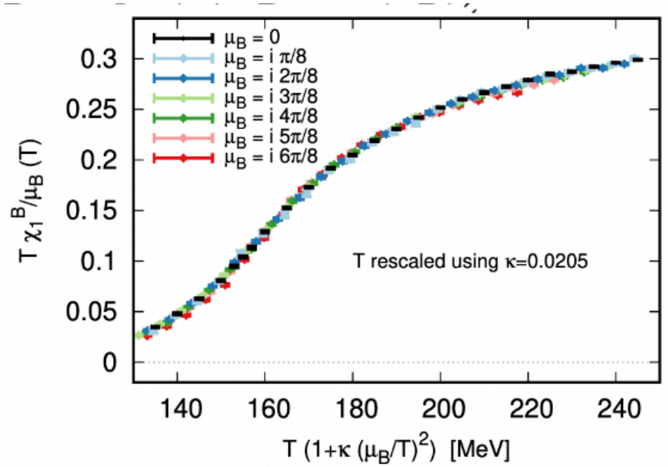

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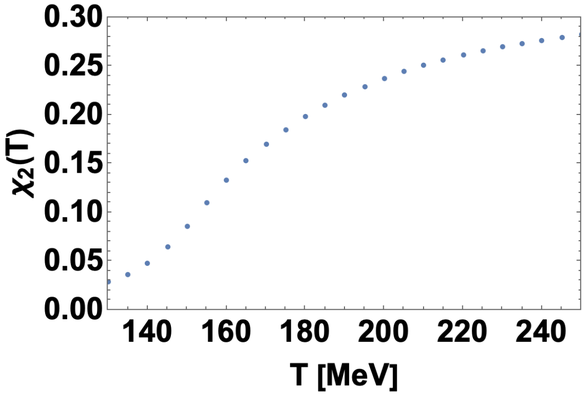

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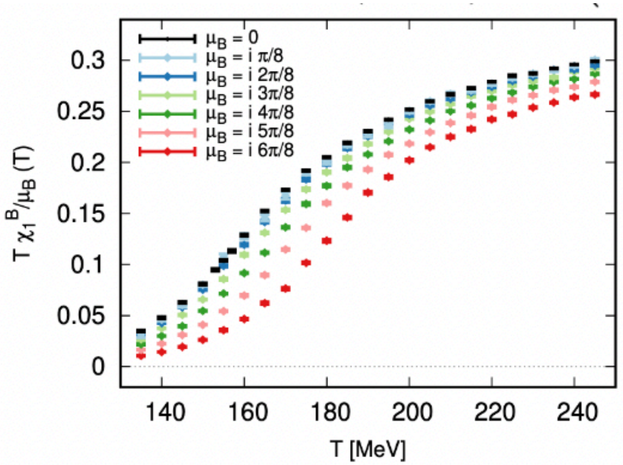

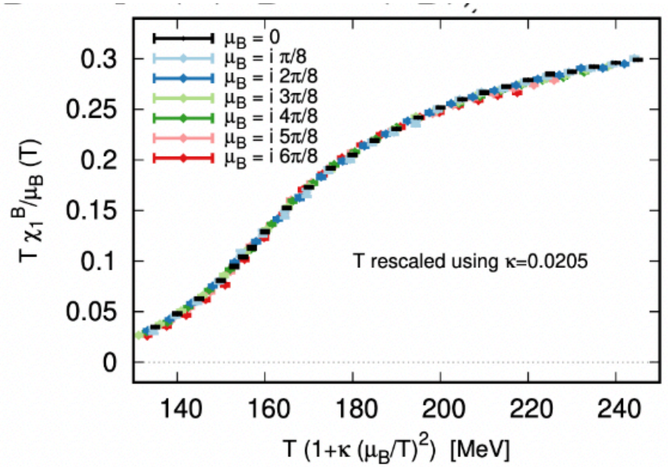

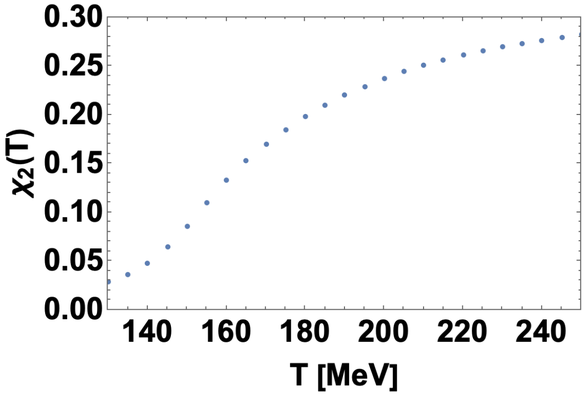

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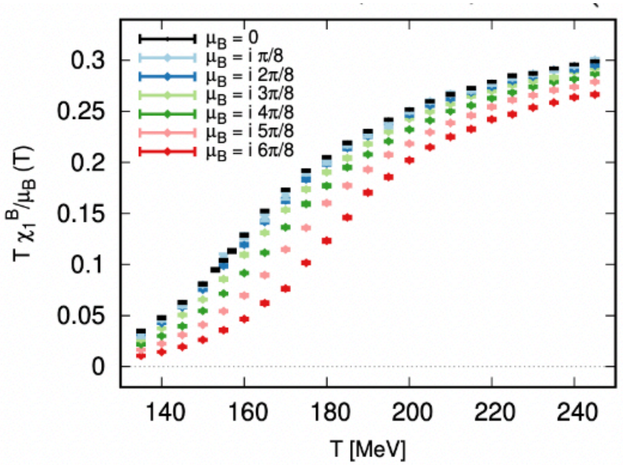

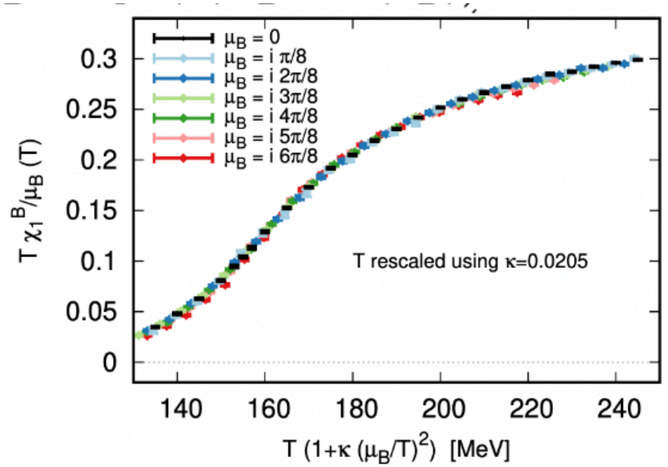

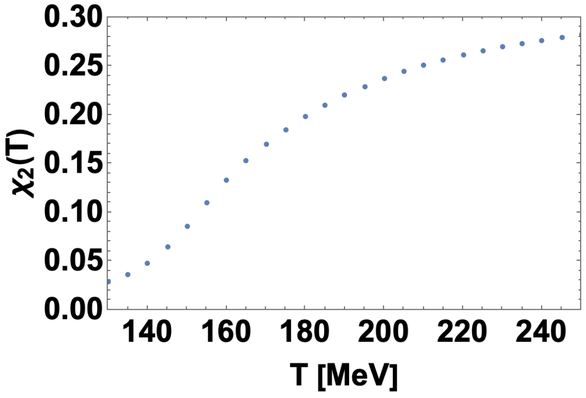

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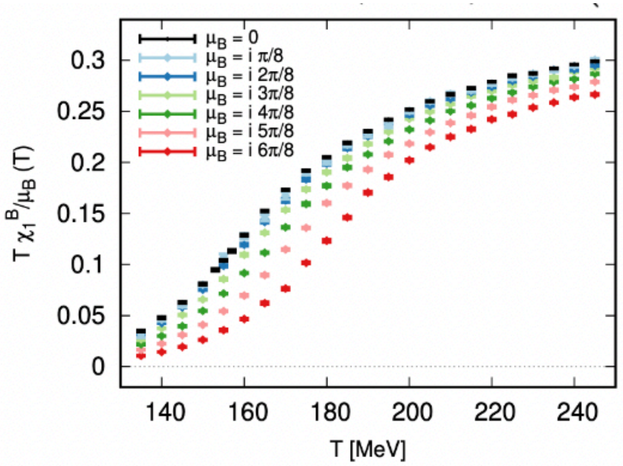

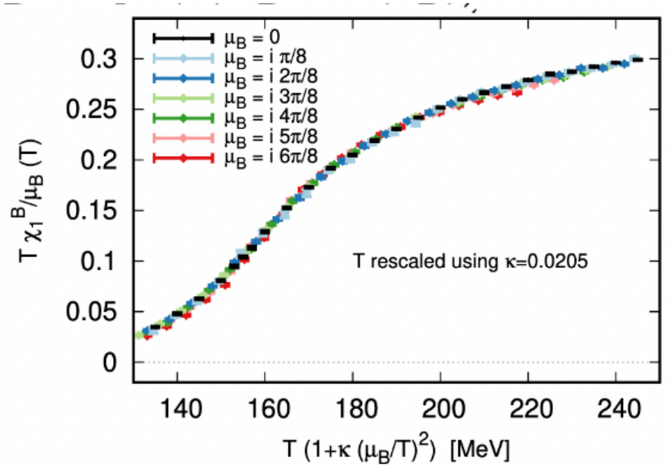

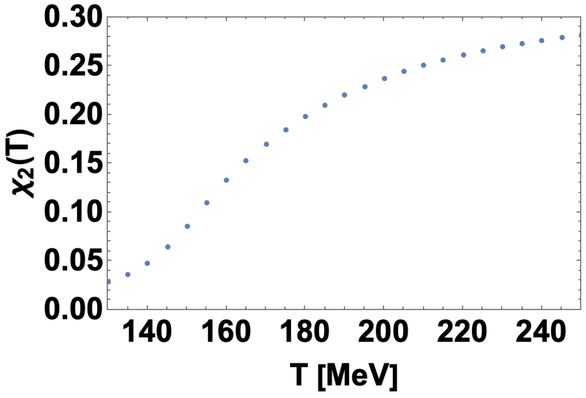

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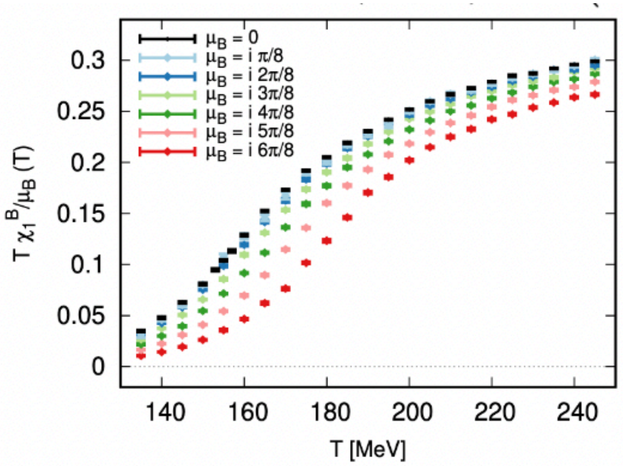

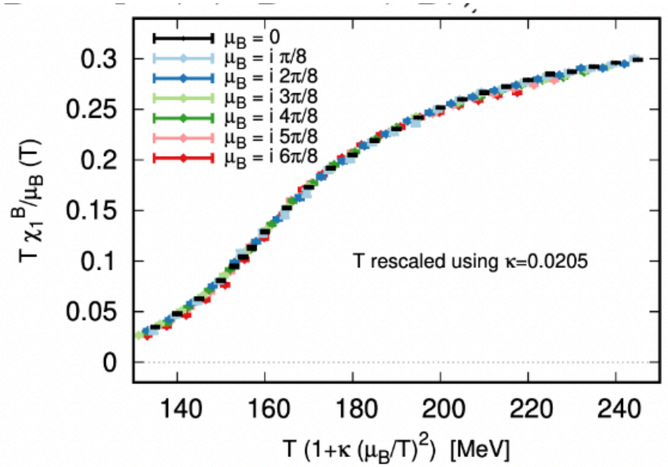

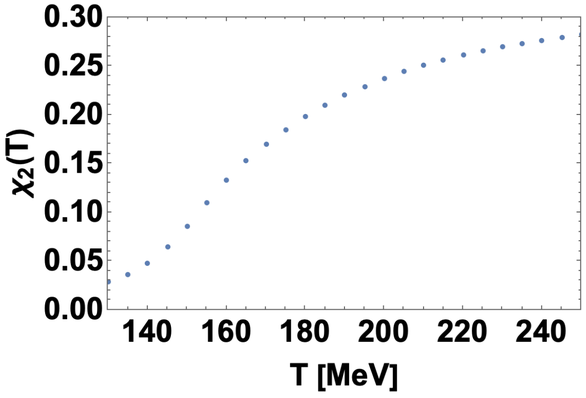

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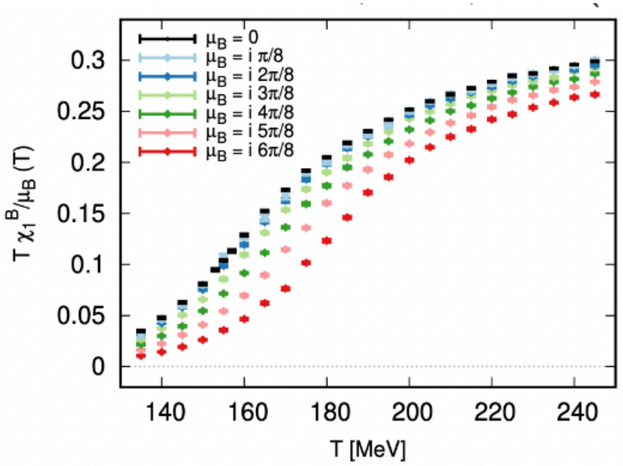
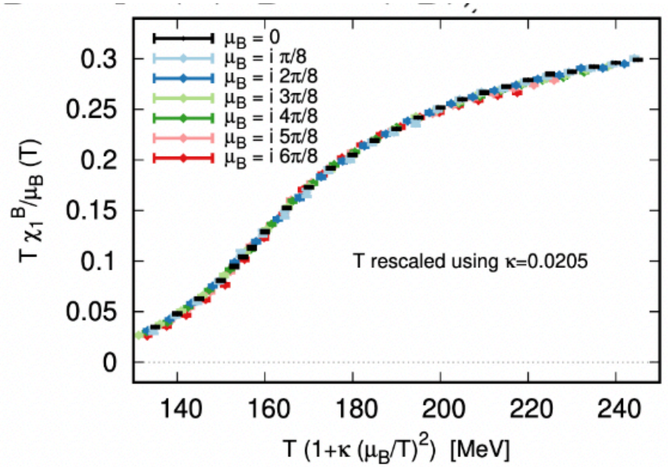
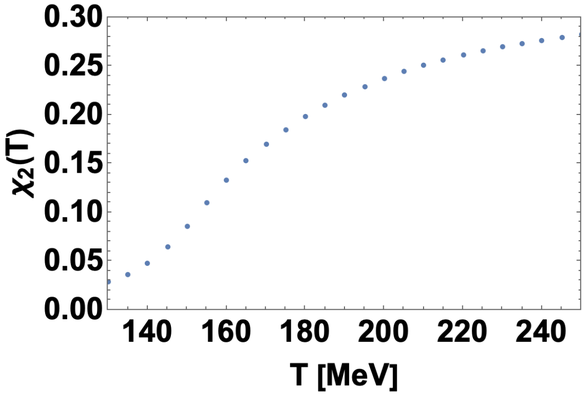
- **μ_B dependence is captured in T-rescaling.**

Alternative Expansion scheme

Simulating at Imaginary μ_B







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- **μ_B dependence is captured in T-rescaling.**
- **Trusted up to** $\frac{\mu_B}{T} = 3.5$

[Borsányi, S. et al. PRL (2021)]

Alternative Expansion scheme

Comparing **Taylor expansion** and **Alternative expansion**

- $\kappa_2^{BB}(T) = \dfrac{1}{6T} \dfrac{\chi_4^B(T)}{\left(\dfrac{\partial \chi_2^B(T)}{\partial T}\right)}$

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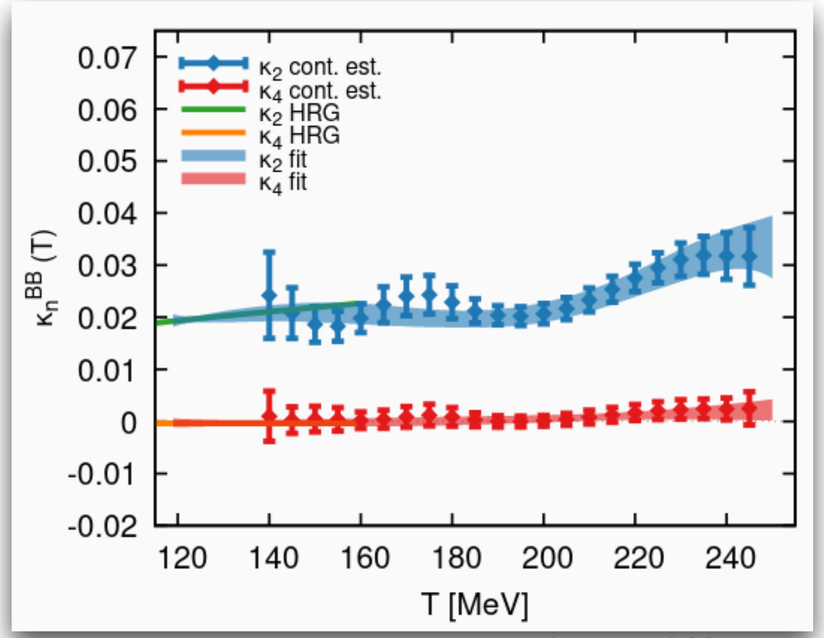

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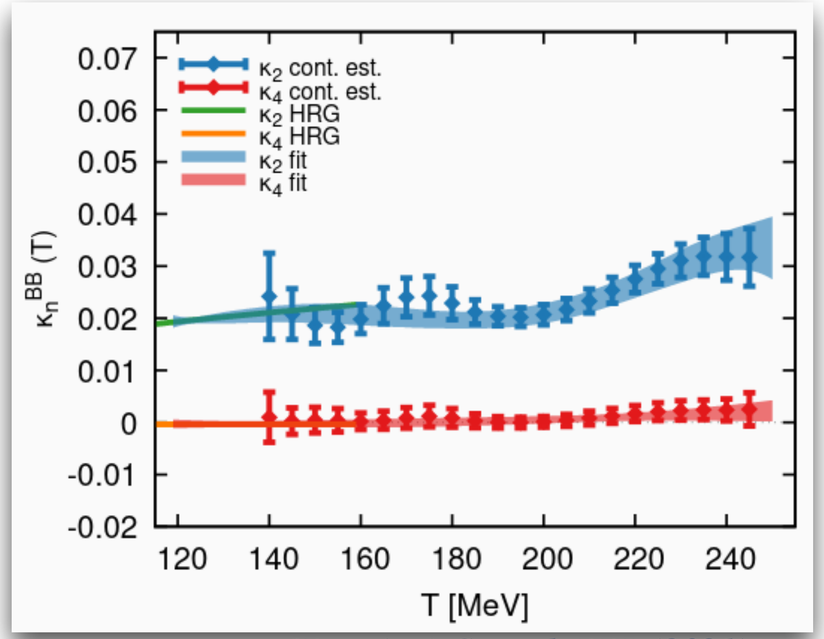

[Borsányi, S. et al. PRL (2021)]

Pros

- $\kappa_2(T)$ **is fairly constant over a large T-Range**
- **There is a separation of scale between** $\kappa_2(T)$ **and** $\kappa_4(T)$
- $\kappa_4(T)$ **is almost zero** $\rightarrow$ **faster convergence**
- **A good agreement with HRG results at Low Temperature**

Introducing Critical Point

3D Ising to QCD Mapping

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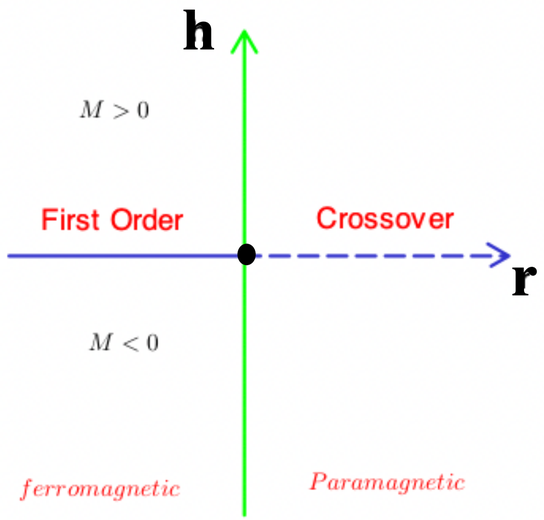
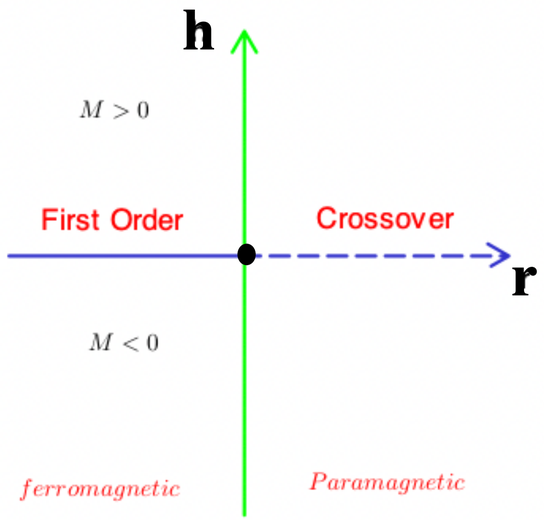
3D Ising to QCD Mapping



3D-Ising Co-ordinates

Introducing Critical Point

3D Ising to QCD Mapping



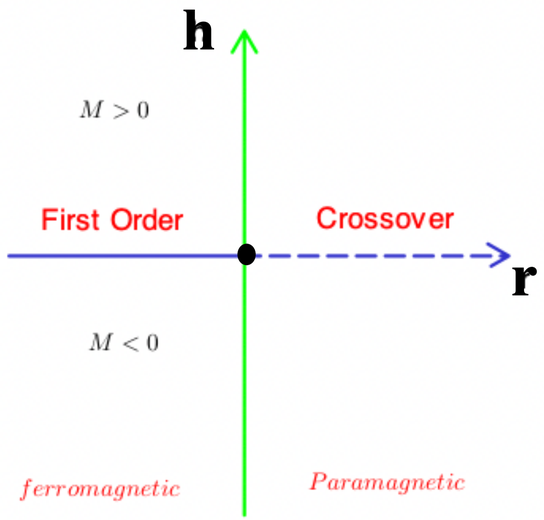
3D-Ising Co-ordinates

$$\frac{T' - T_0}{T_0} = - w\, h \sin \alpha_{12}$$

$$\frac{\mu_B^2 - \mu_{BC}^2}{T_0^2} = w(-r\rho - h\cos \alpha_{12})$$

Introducing Critical Point

3D Ising to QCD Mapping



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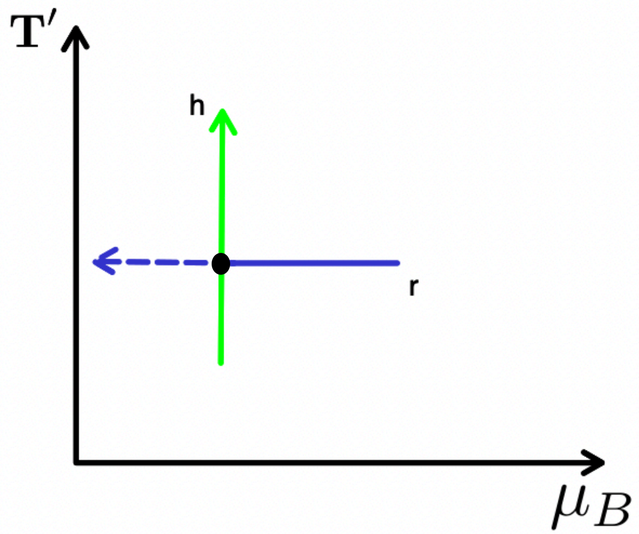

(T', μ_B)-Co-ordinates

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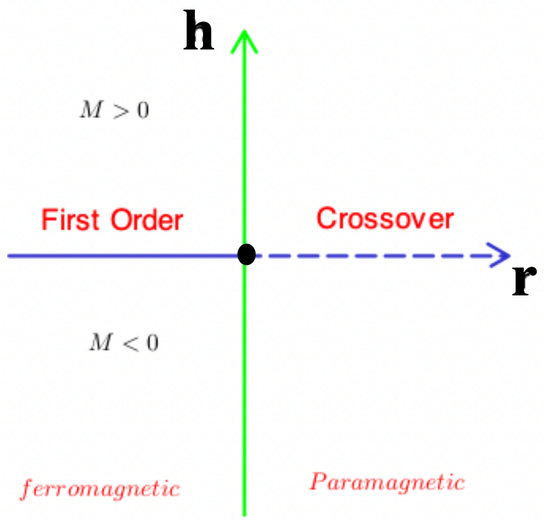

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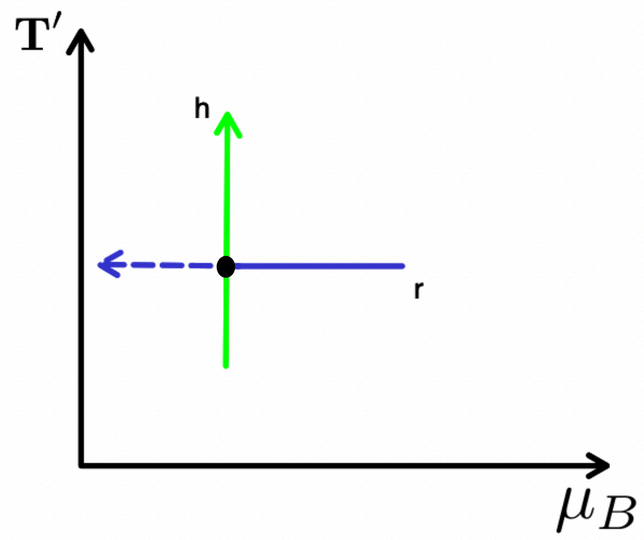
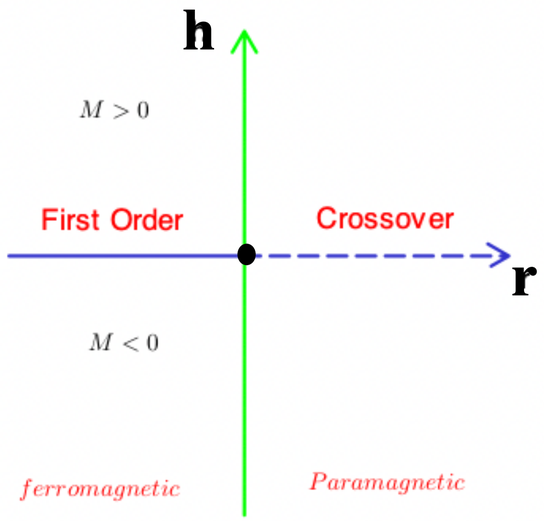


(T′, μ_B)-Co-ordinates

$$T' = T\left[1 + \left(\frac{\mu_B}{T}\right)^2 \kappa_2^{BB}(T) + \mathcal{O}\left(\frac{\mu_B}{T}\right)^4\right]$$

Introducing Critical Point

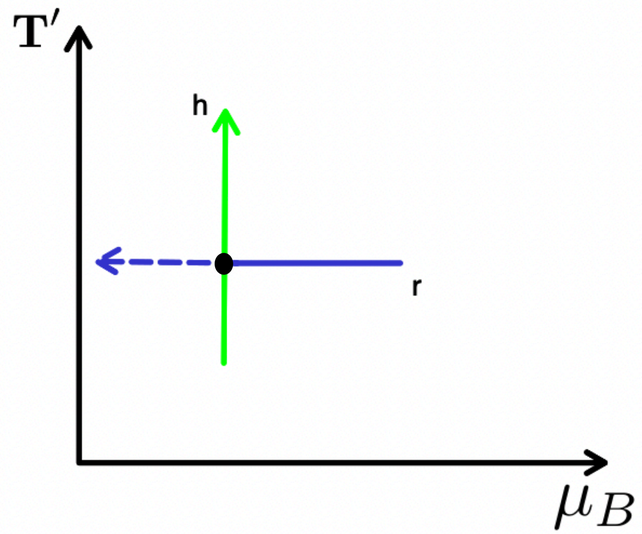
3D Ising to QCD Mapping



3D-Ising Co-ordinates

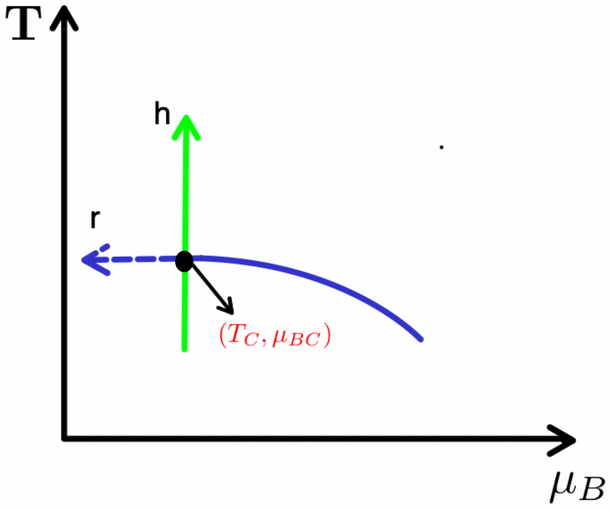
$$\frac{T' - T_0}{T_0} = -\, w\, h \sin\, \alpha_{12}$$

$$\frac{\mu_B^2 - \mu_{BC}^2}{T_0^2} = w(-r\rho - h\cos\, \alpha_{12})$$



(T′, μ_B)-Co-ordinates

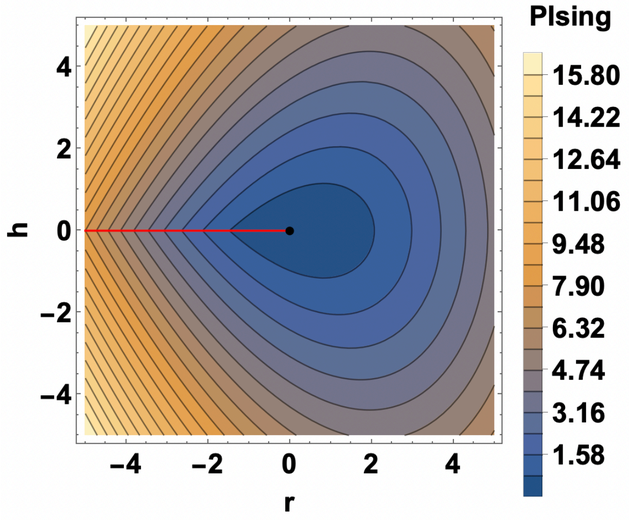
$$T' = T\left[1 + \left(\frac{\mu_B}{T}\right)^2 \kappa_2^{BB}(T) + \mathcal{O}\left(\frac{\mu_B}{T}\right)^4\right]$$



QCD - Co-ordinates

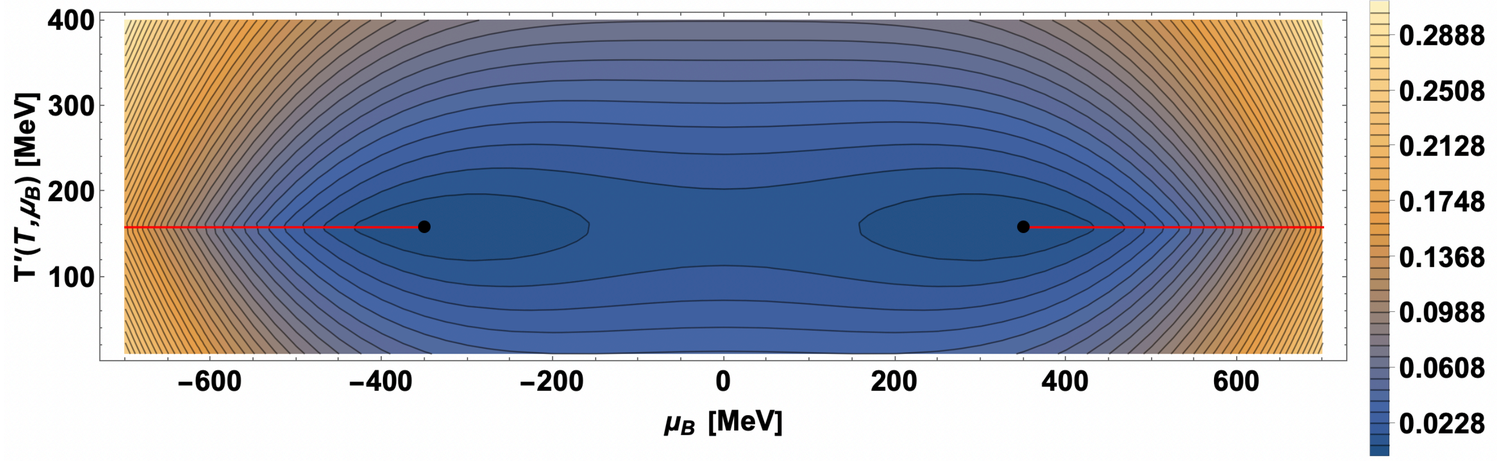
Introducing Critical Point

3D Ising to QCD Mapping



3D-Ising coordinates

Ising Pressure



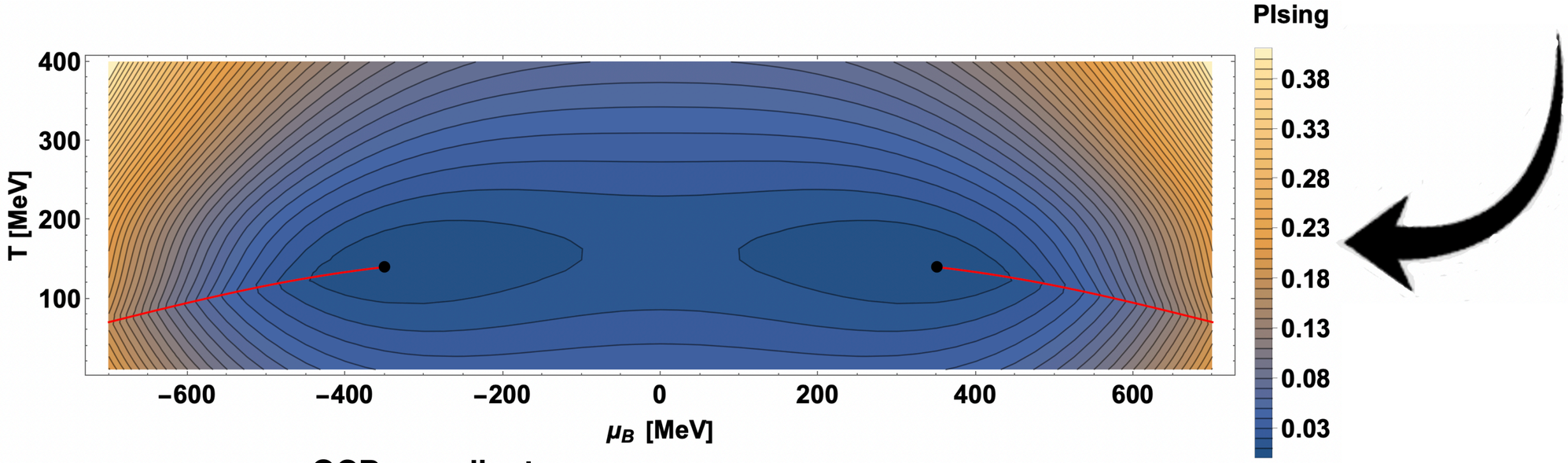
T'-coordinates

Parameters

$w = 10,\ \rho = 0.5$

$\mu_{BC} = 350\ MeV,\ T_0 = 158\ [MeV]$

$$T_C\left[1+\kappa_2(T)\left(\frac{\mu_B}{T_C}\right)^2\right] = T_0$$

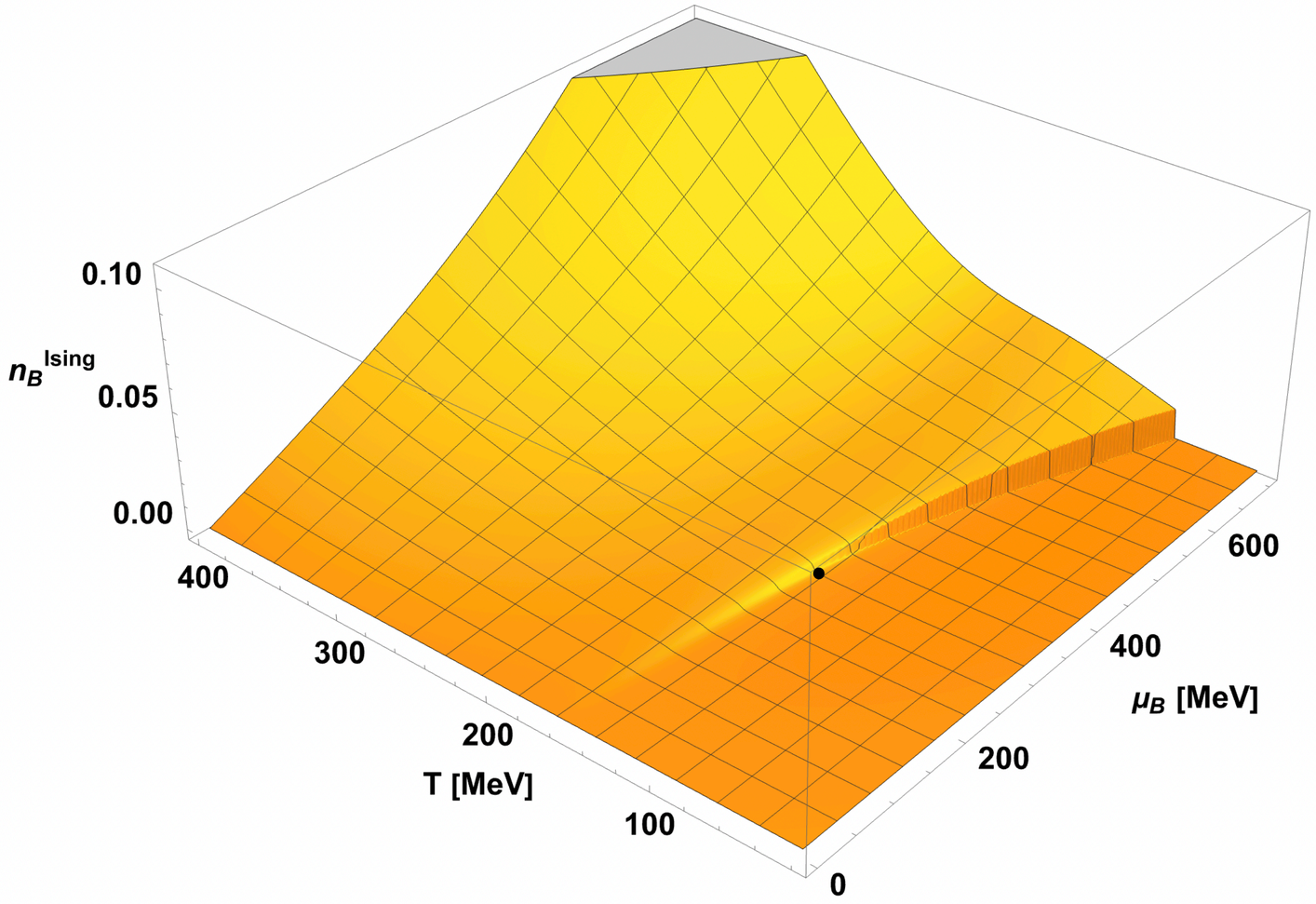


QCD coordinates

Introducing Critical Point

Ising Baryon Density

$$\chi_1^{Ising}(T,\mu_B) = n_B^{Ising}(T,\mu_B) = \frac{\partial(P^{Ising}(T,\mu_B)/T^4)}{\partial(\mu_B/T)}$$



Re-Constructing the Full Baryon Density

Re-Constructing the Full Baryon Density

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi_{2,Lattice}^{B}(T'_{full}(T,\mu_B),0)$$

Re-Constructing the Full Baryon Density

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi^B_{2,Lattice}(T'_{full}(T,\mu_B),0)$$

$$T'_{full}(T,\mu_B) = \underbrace{T'_{Lattice}(T,\mu_B)}_{\text{lowest order in } (\frac{\mu_B}{T})} + \underbrace{T'_{crit}(T,\mu_B) - Taylor[T'_{crit}(T,\mu_B)]}_{\text{higher orders in } (\frac{\mu_B}{T})}$$

Lattice Term

Ising Term

Re-Constructing the Full Baryon Density

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi^B_{2,Lattice}(T'_{full}(T,\mu_B),0)$$

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Introducing a Critical Point

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Lattice Term

Ising Term

Introducing a Critical Point

$$T'_{crit}(T,\mu_B) \approx T_0 + \left(\left.\frac{\partial\chi^B_{2,lattice}(T,0)}{\partial T}\right|_{T=T_0}\right)^{-1} \frac{n_B^{crit}(T,\mu_B)}{T^3(\mu_B/T)} + \ldots$$

Re-Constructing the Full Baryon Density

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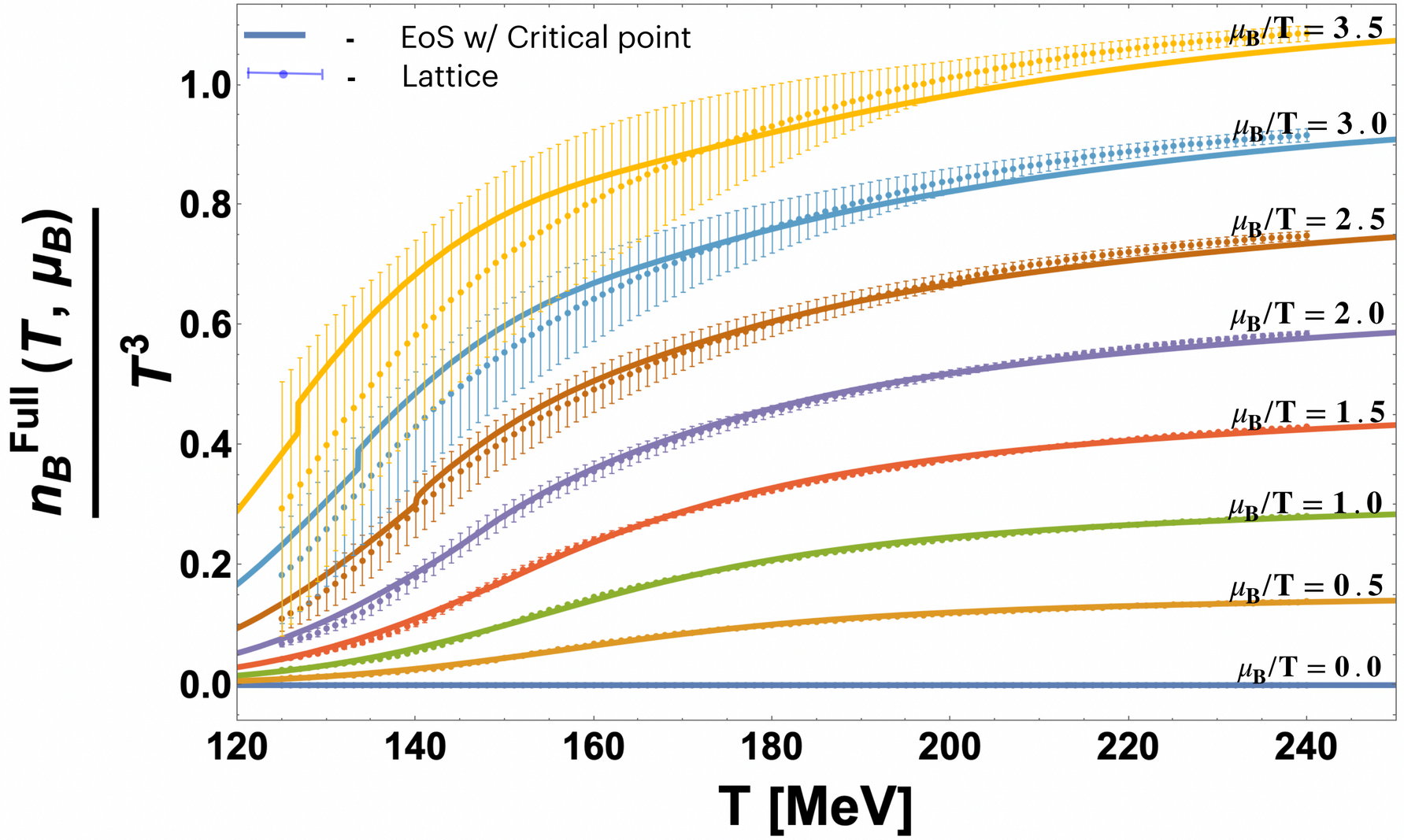
Baryon density results

Full Baryon Density at a constant $\frac{\mu_B}{T}$ compared with Lattice

$\mu_B = 350$ [MeV], $\alpha_{12} = 90$, $\rho = 2$, $w = 2$



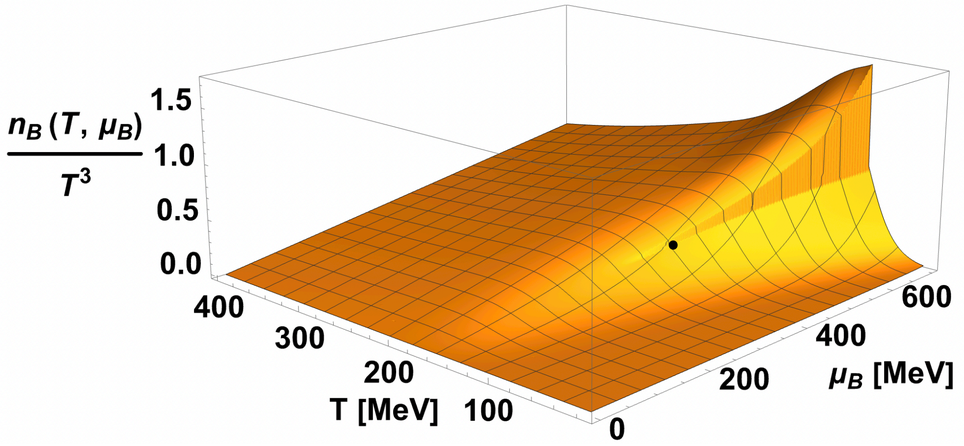
[Borsányi, S. et al. PRL (2021)]

Thermodynamic observables

Thermodynamic observables

Ising-AltExS

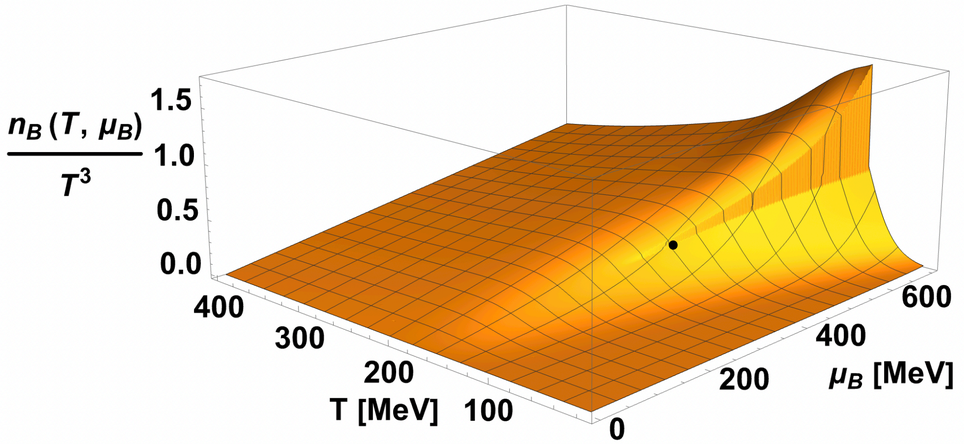
Baryon Density



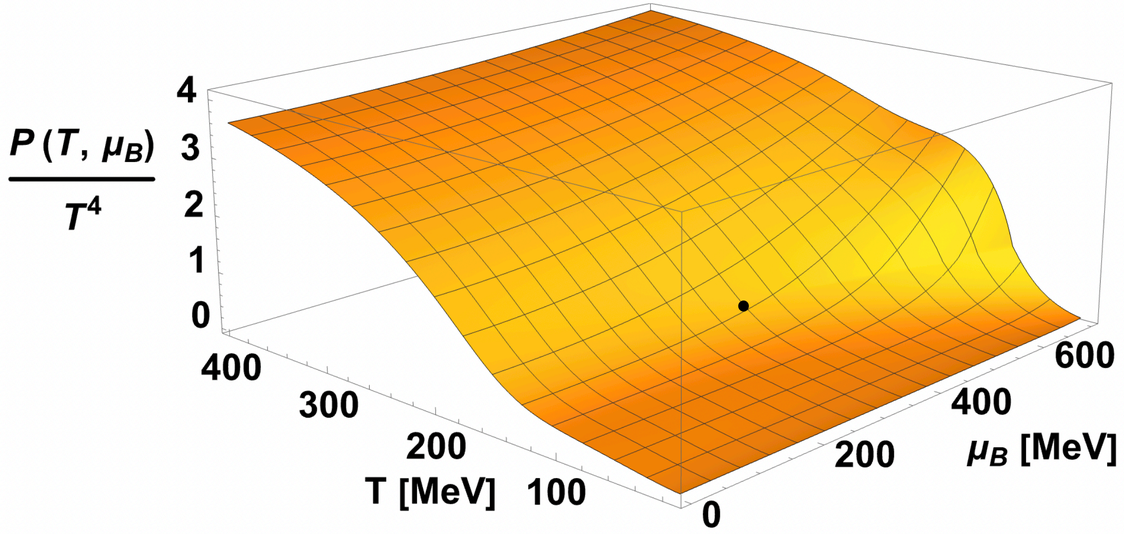
Thermodynamic observables

Ising-AltExS

Baryon Density



Pressure



Thermodynamic observables

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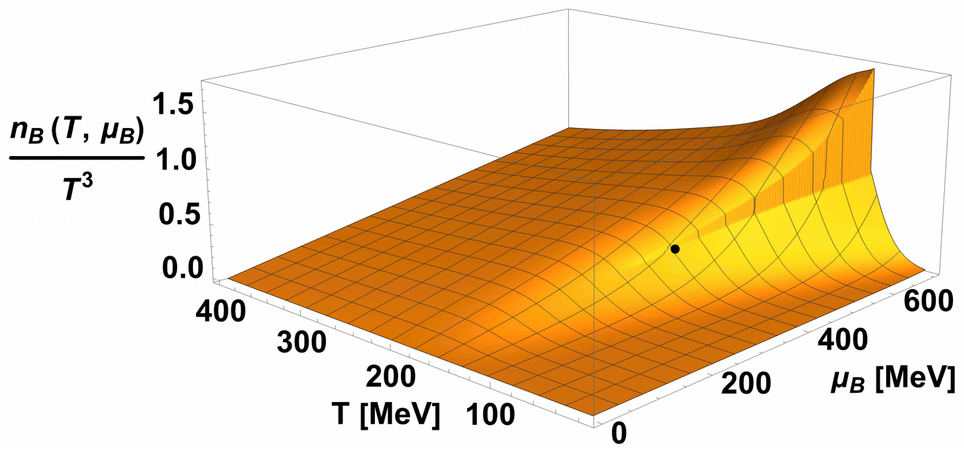

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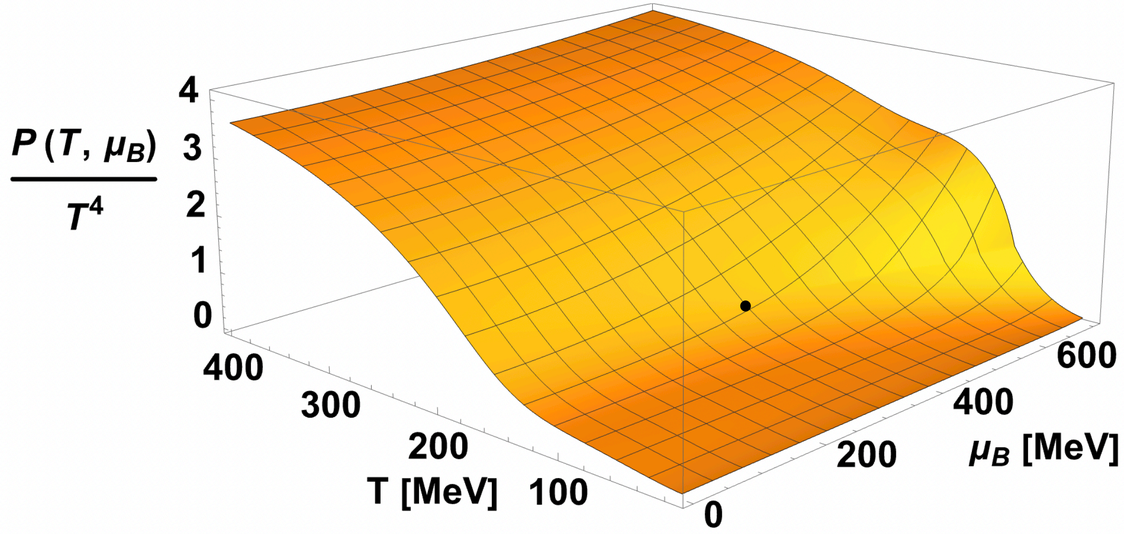
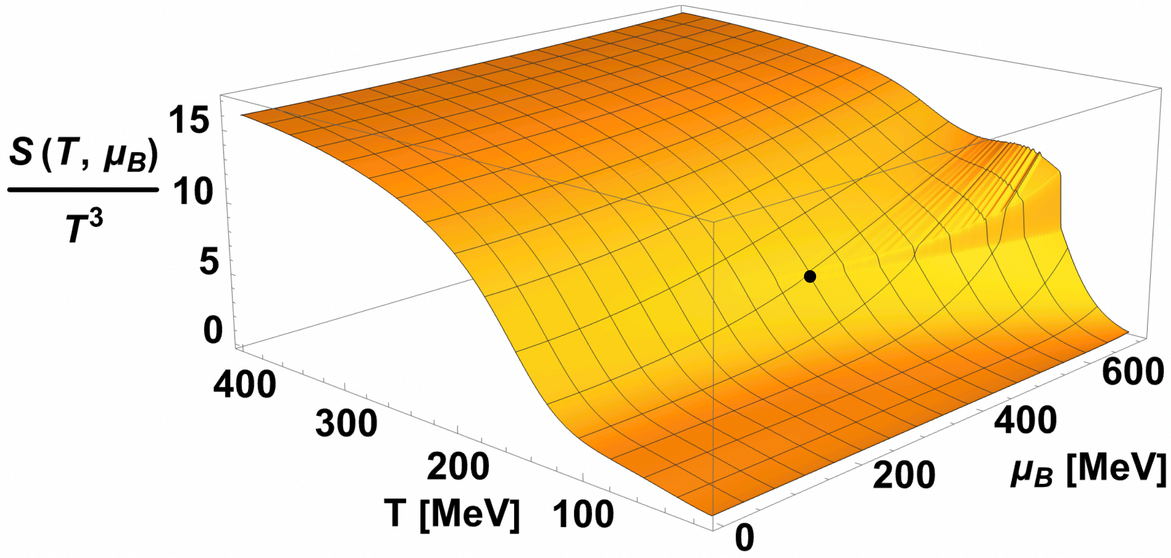


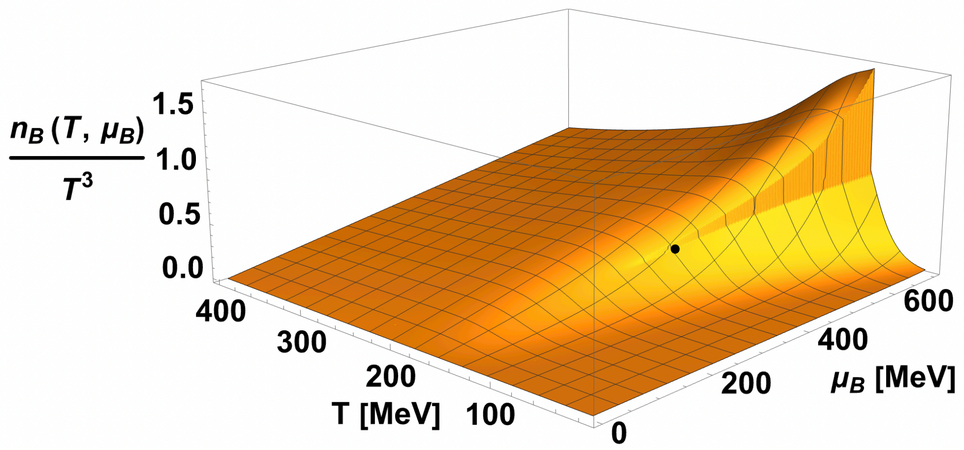
Entropy Density



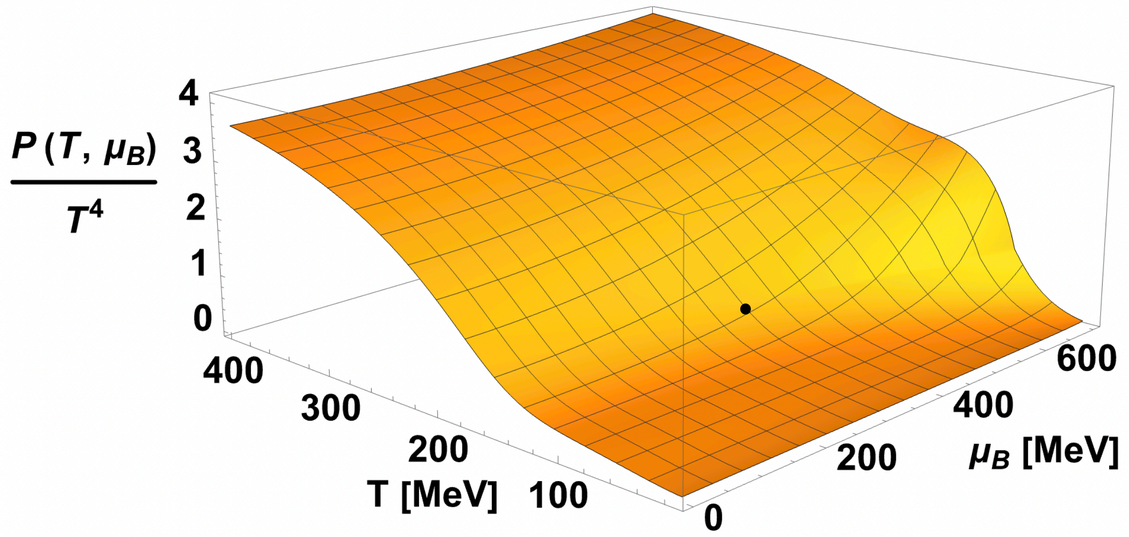
Thermodynamic observables

Ising-AltExS

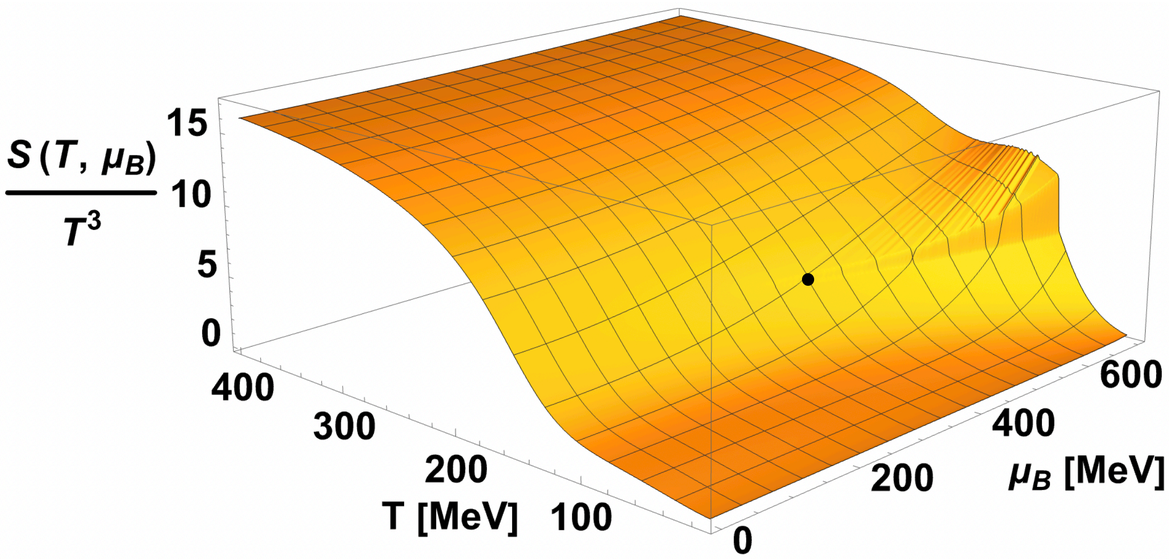
Baryon Density



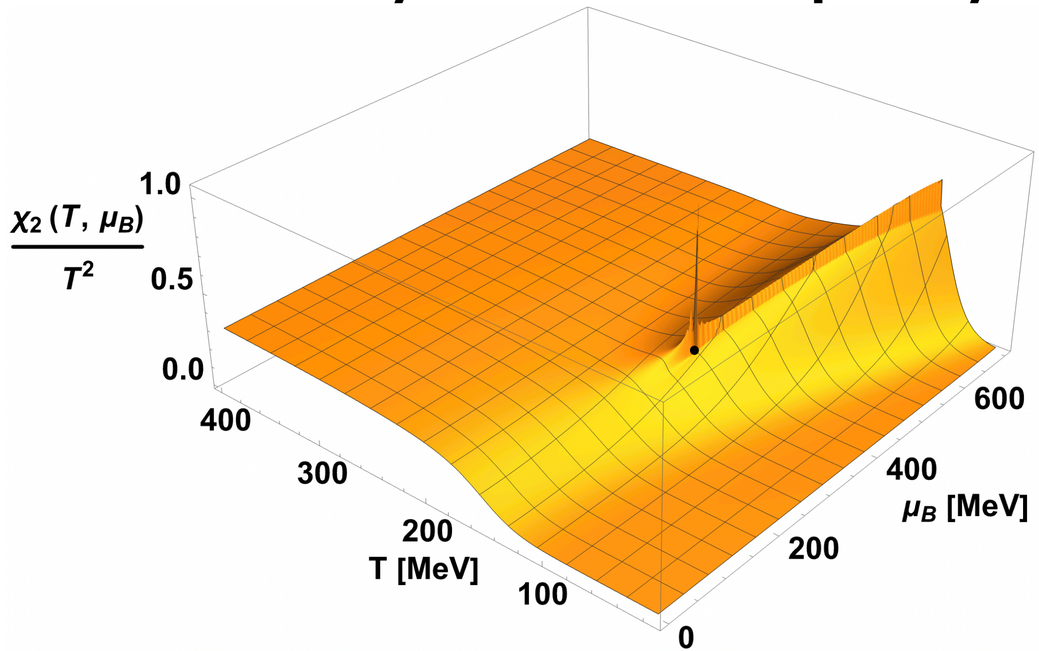
Pressure



Entropy Density



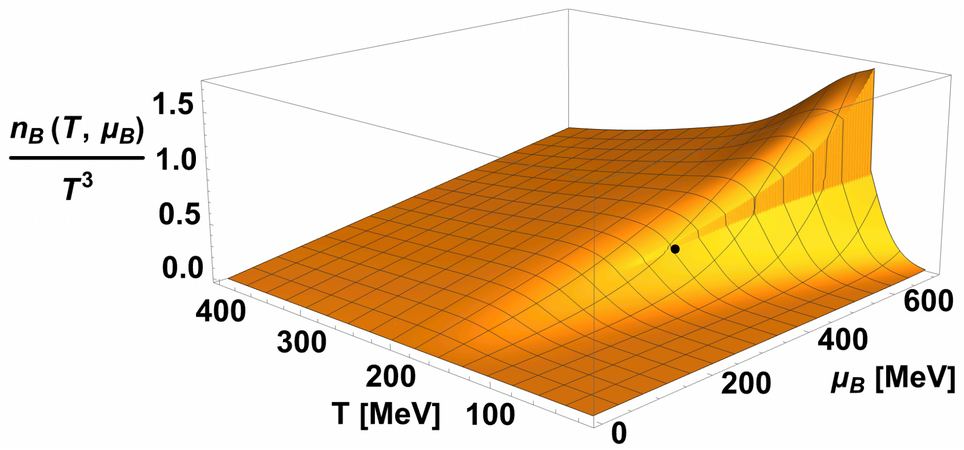
Baryon number Susceptibility



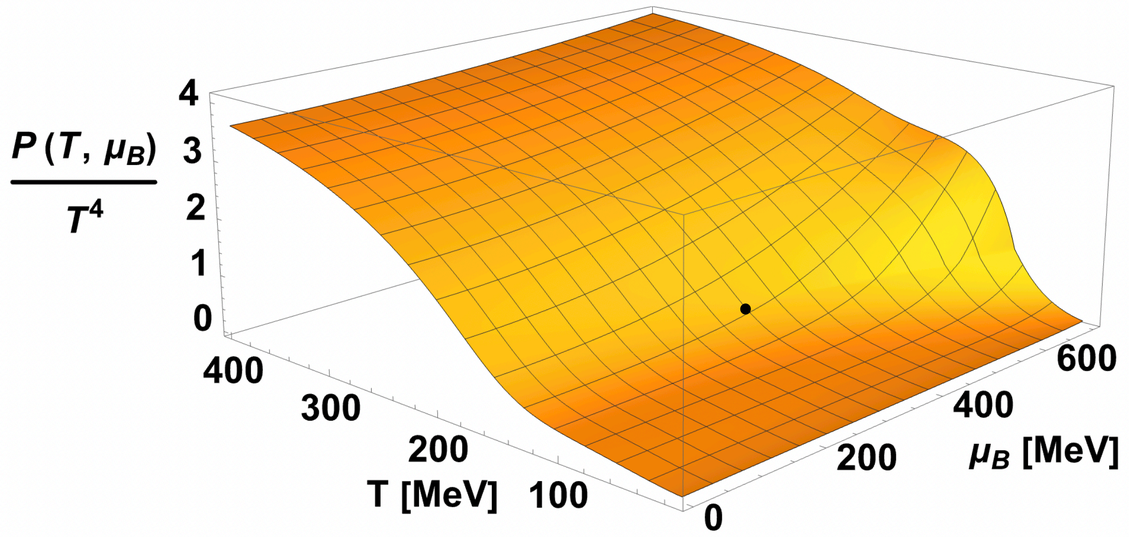
Thermodynamic observables

Ising-AltExS

Baryon Density



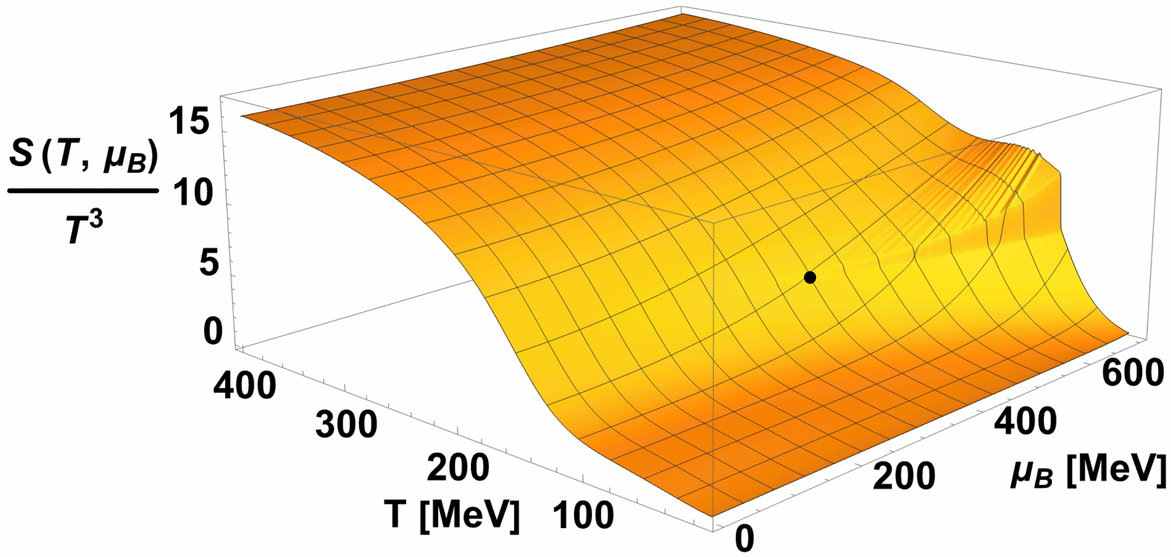
Pressure



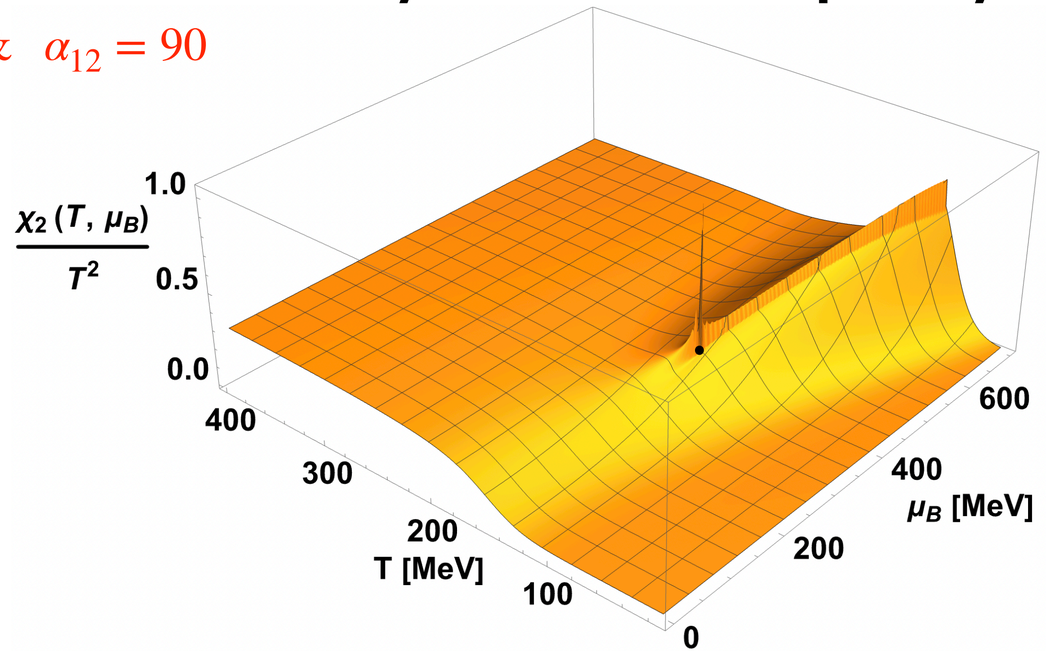
- **Critical Point**

$w = 2 \quad , \quad \rho = 2 \quad \& \quad \alpha_{12} = 90$

Entropy Density



Baryon number Susceptibility



Constraints of our EoS

Constraints of our EoS

Physical Quark masses

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Physical Quark masses

- Relates the parameters of the Ising -to-QCD mapping to the quark masses m_q
- Suggests the angle α_{12} between $r = 0$ and $h = 0$ lines in (T, μ_B) vanishes as $m_q^{2/5}$

[Pradeep, M. S., & Stephanov, M PhysRevD . 100(5), 056003.(2019)]

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Thermodynamic Stability & Causality

Constraints of our EoS

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  [Pradeep, M. S., & Stephanov, M PhysRevD . 100(5), 056003.(2019)]

Thermodynamic Stability & Causality

- Stability $$c_v = \left(\frac{\partial s}{\partial T}\right)\Big|_{n_B} > 0 \qquad \chi_T(T, \mu_B) = \left(\frac{\partial n_B}{\partial \mu_B}\right)\Big|_T = \left(\frac{\partial^2 P}{\partial \mu_B^2}\right)\Big|_T > 0$$

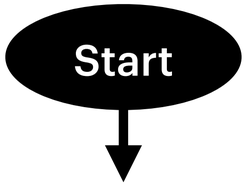
- Causality $$c_s^2(T, \mu_B) < 1$$

Code update !

Code structure

Code structure

Code structure



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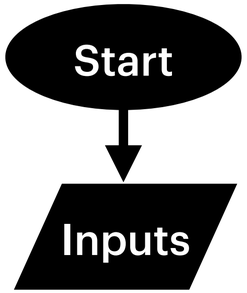

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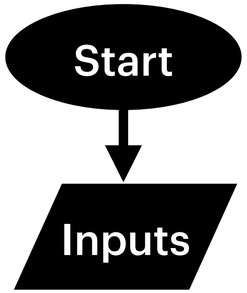


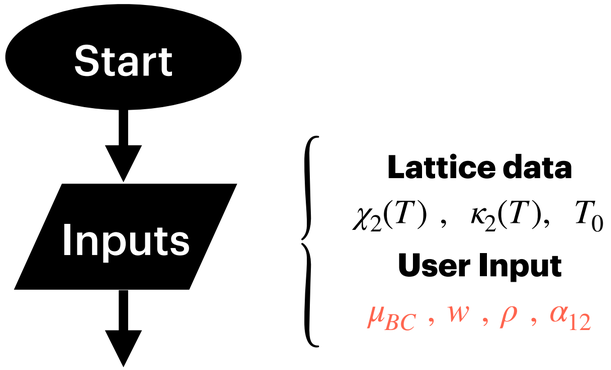
Lattice data

$\chi_2(T)$, $\kappa_2(T)$, T_0

User Input

μ_{BC} , w , ρ , α_{12}

Code structure



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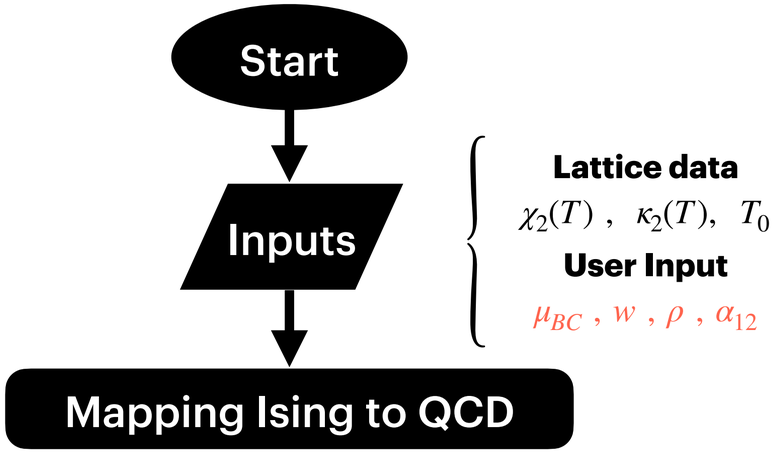

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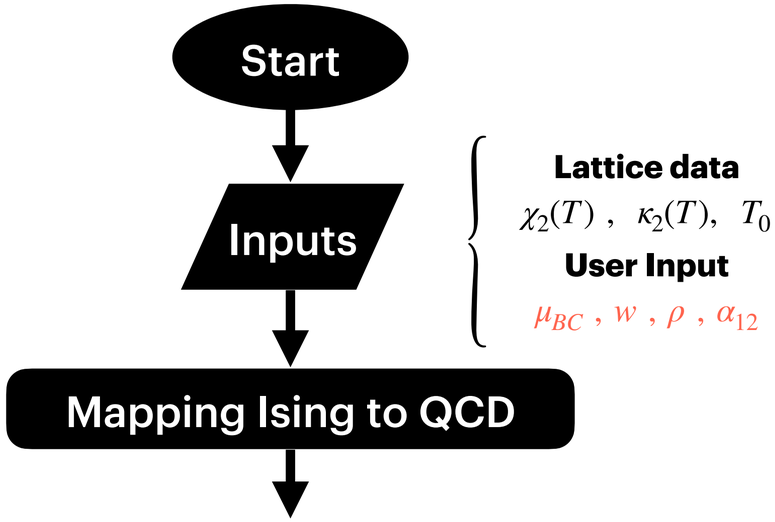

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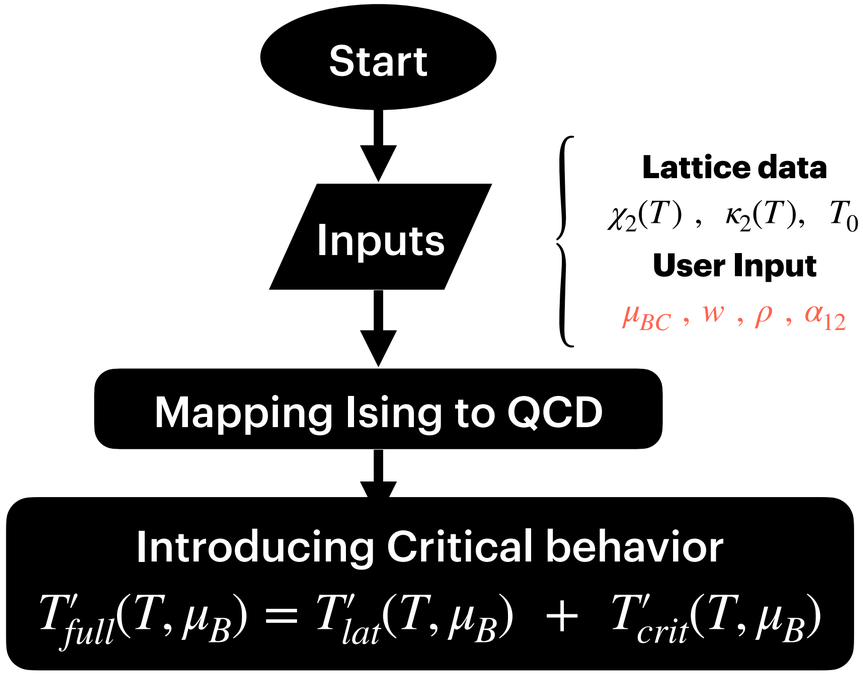

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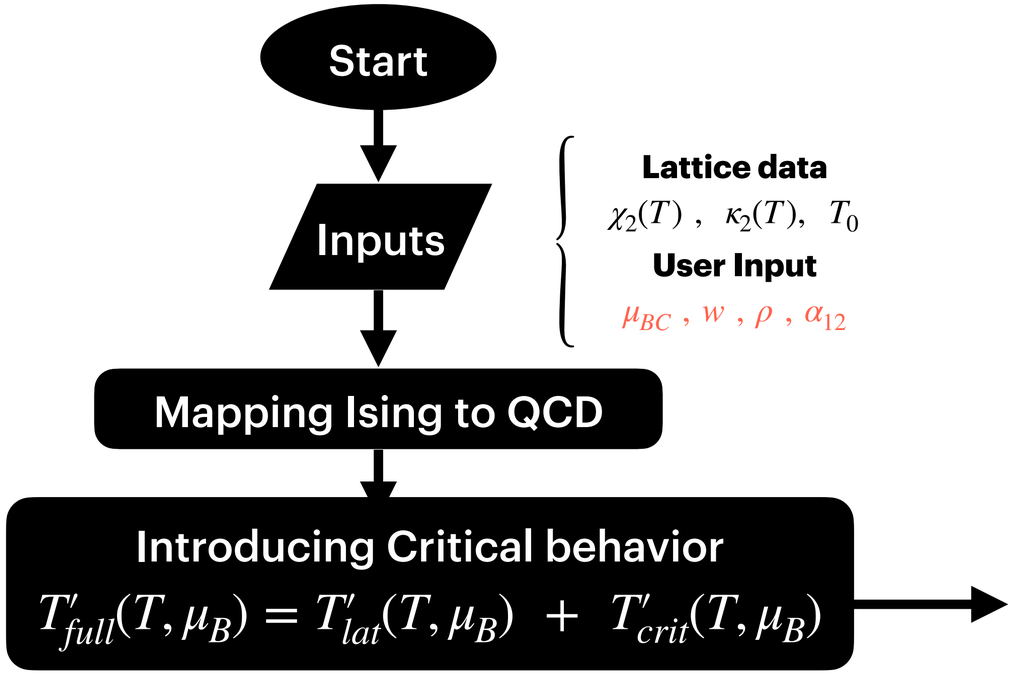

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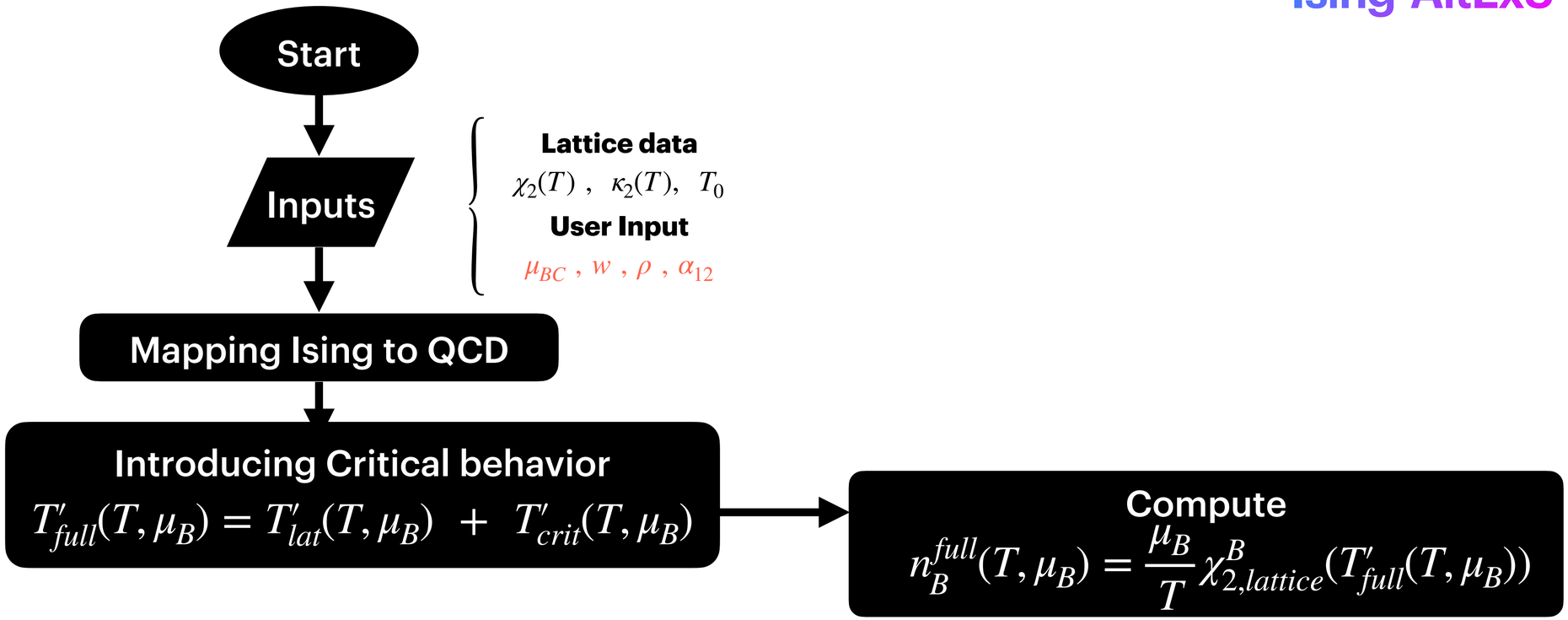

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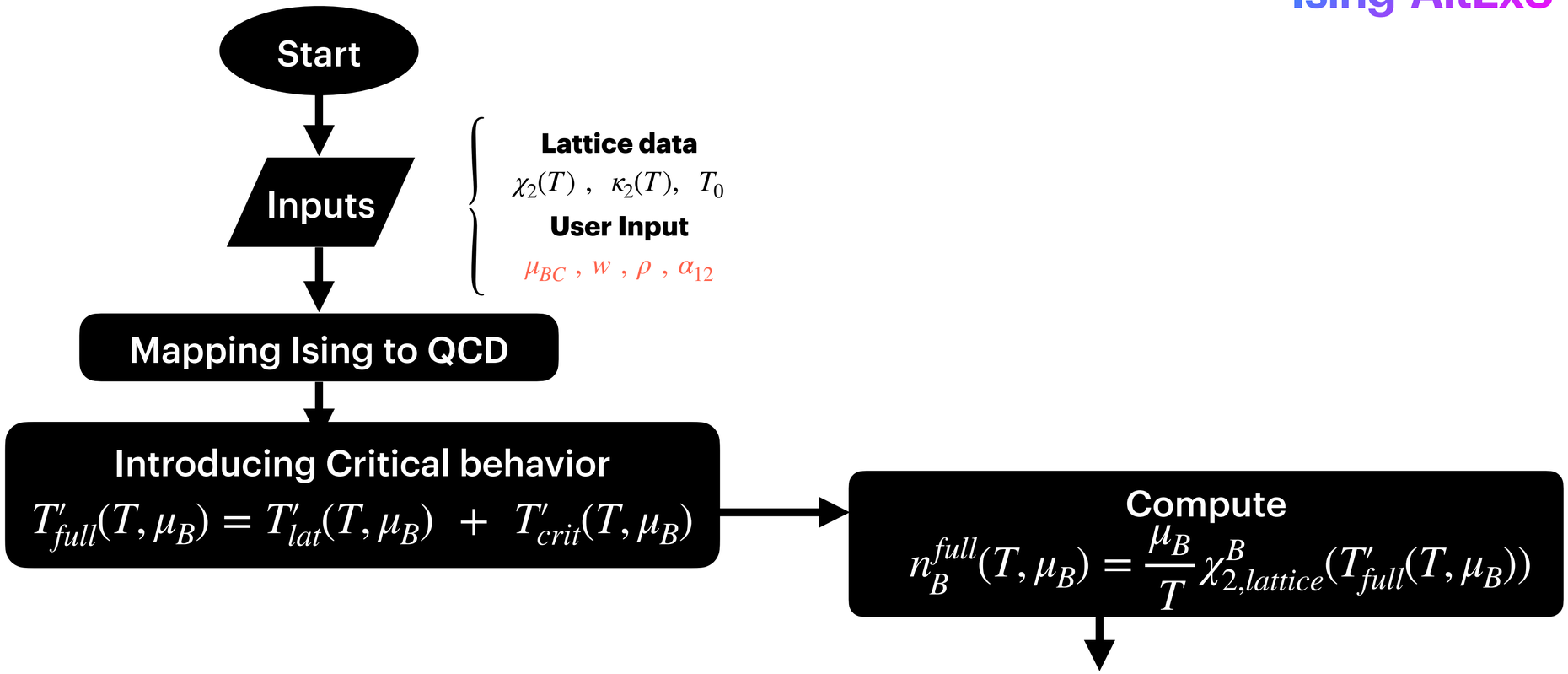

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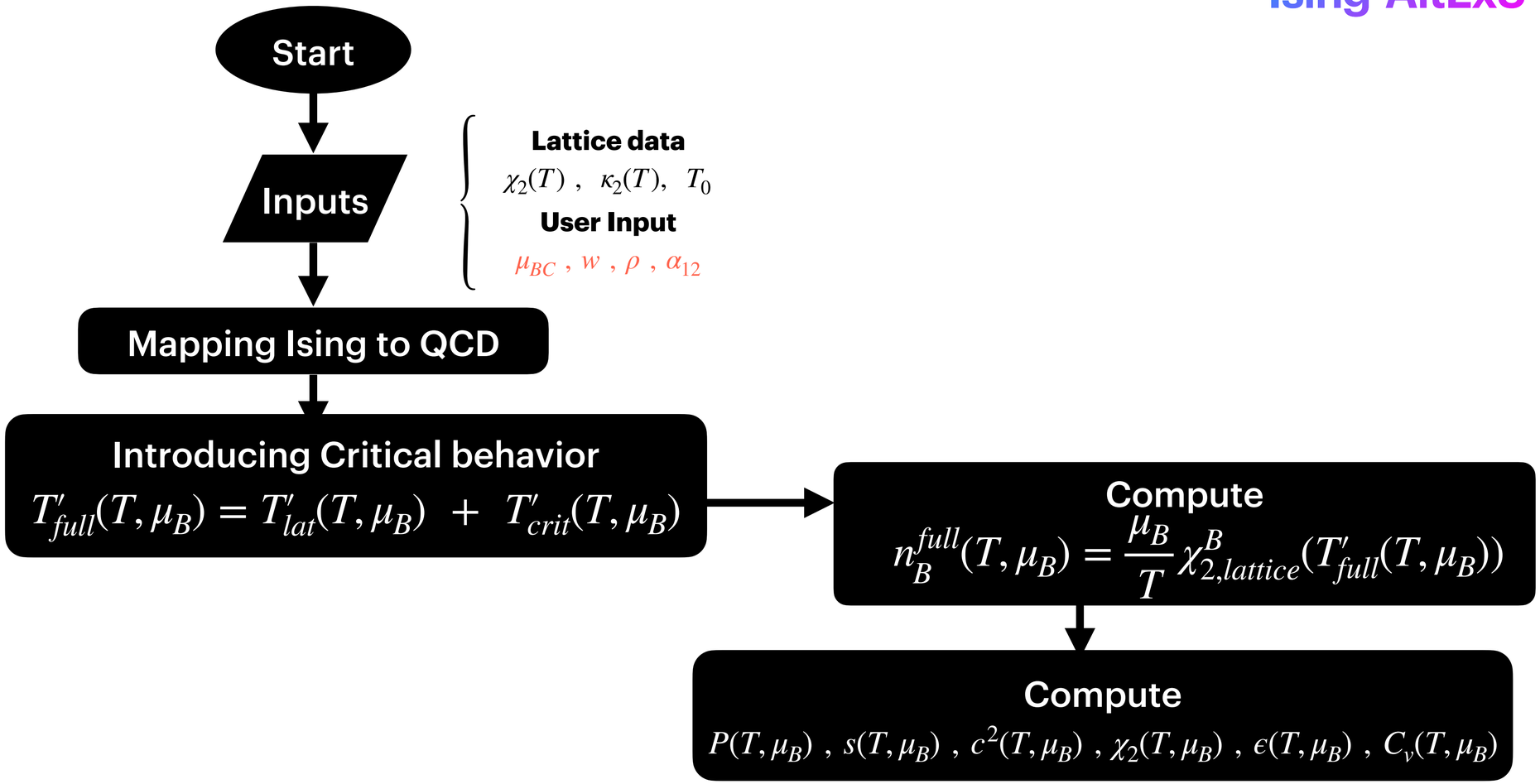

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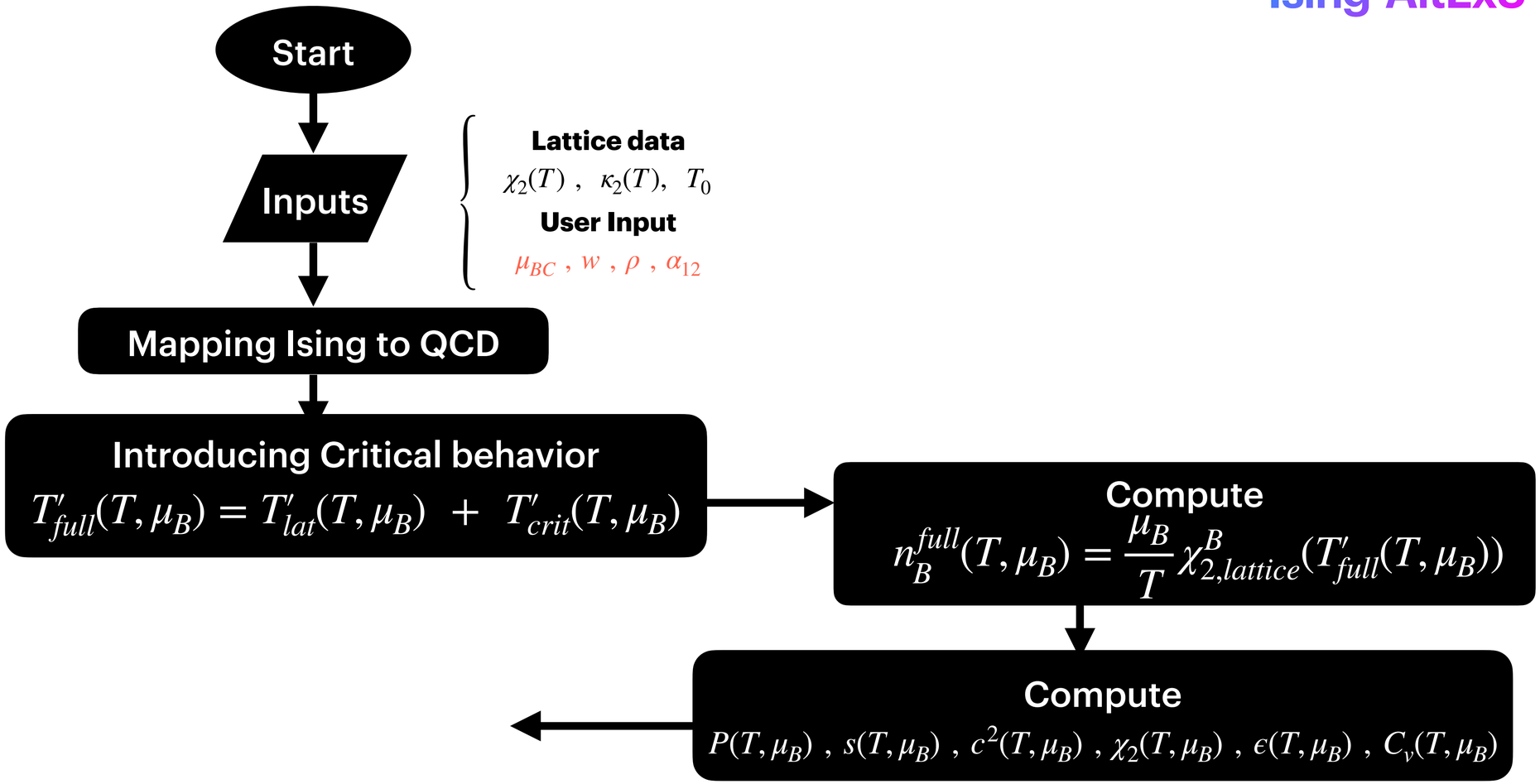

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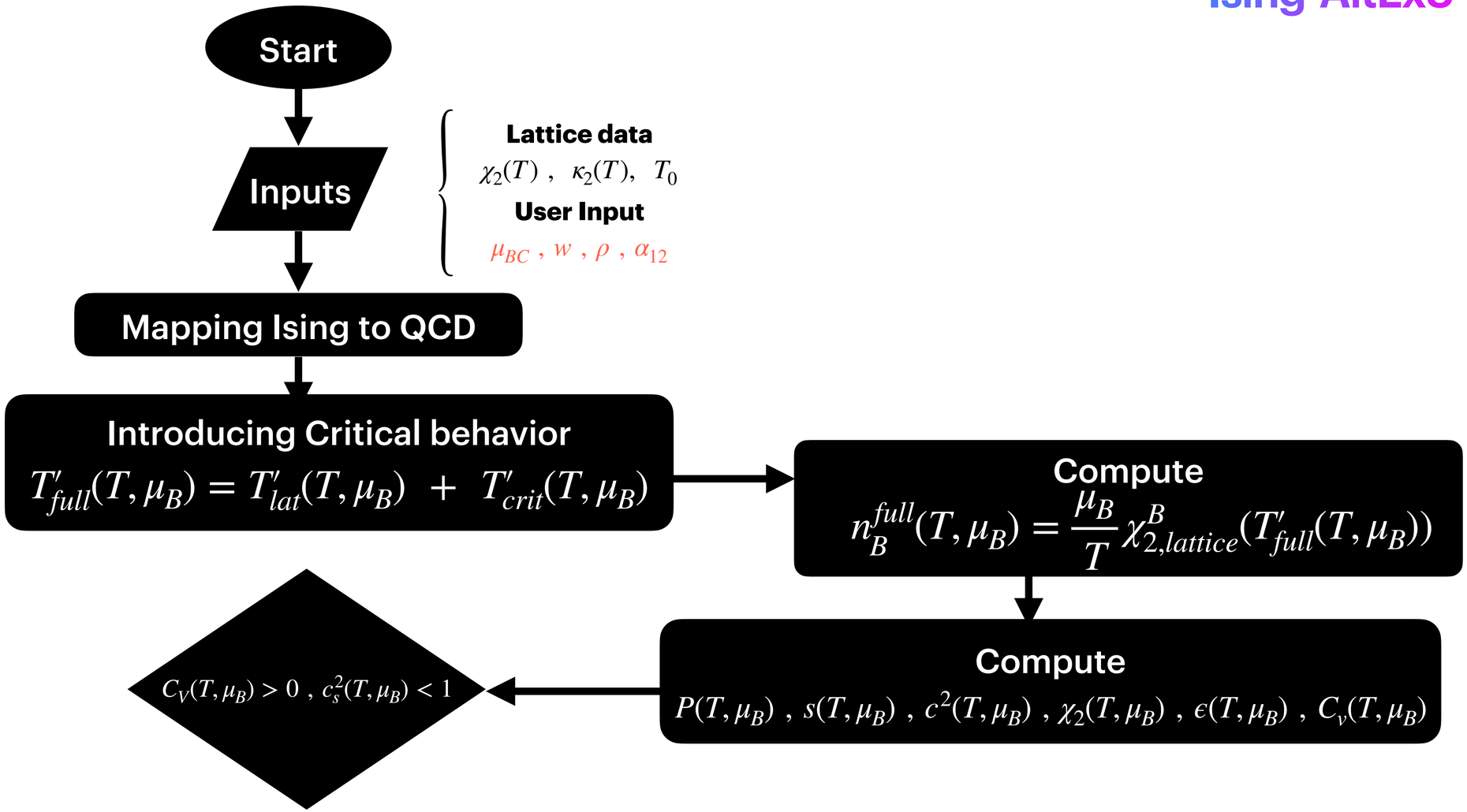

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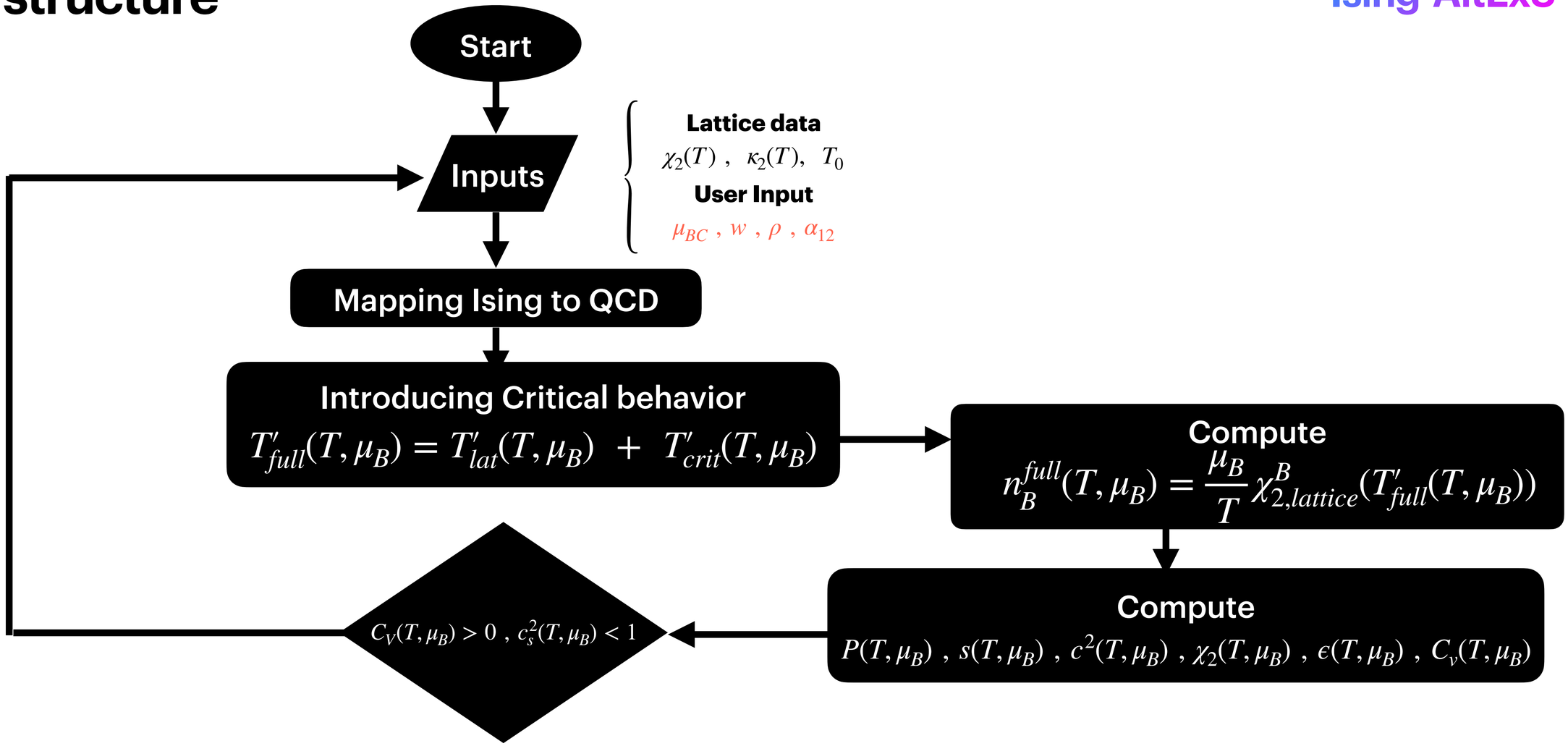

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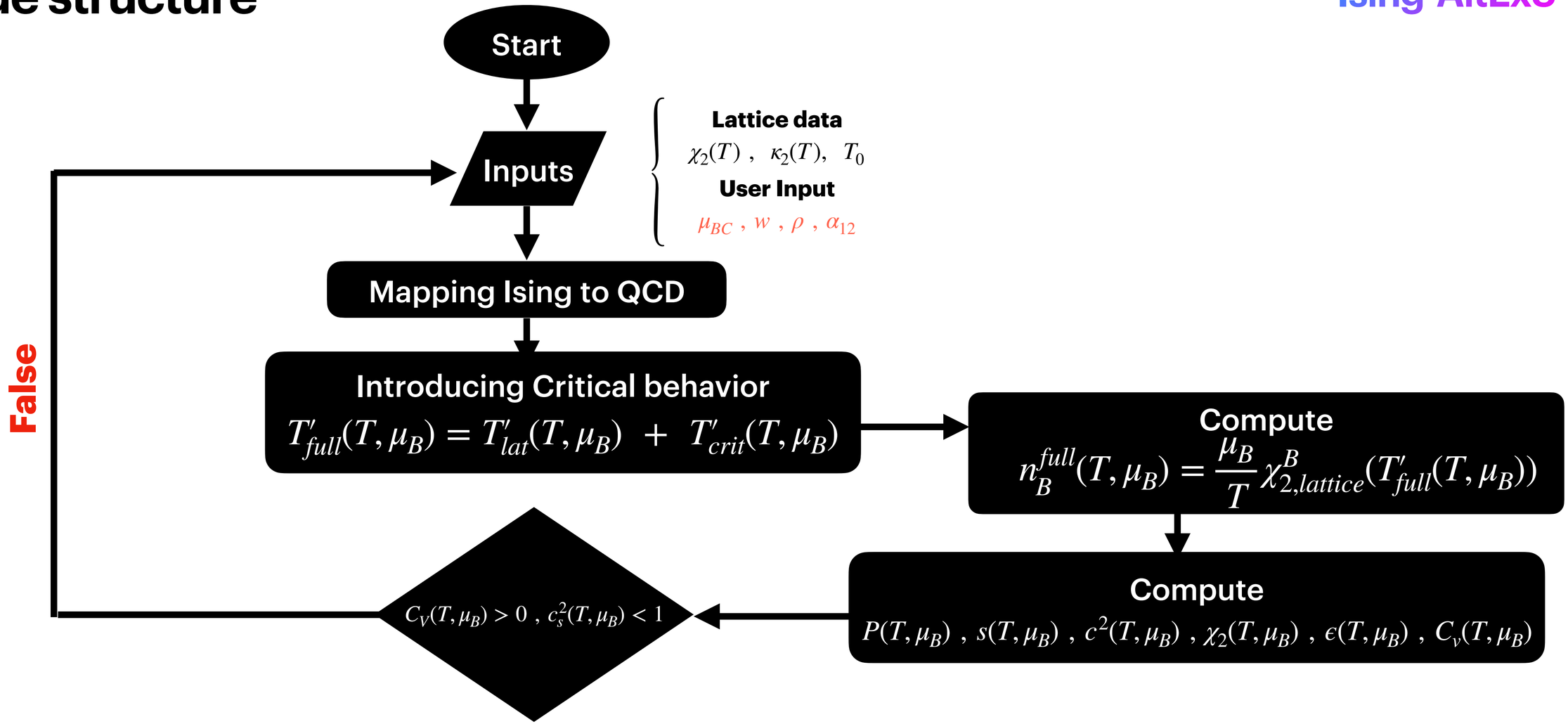
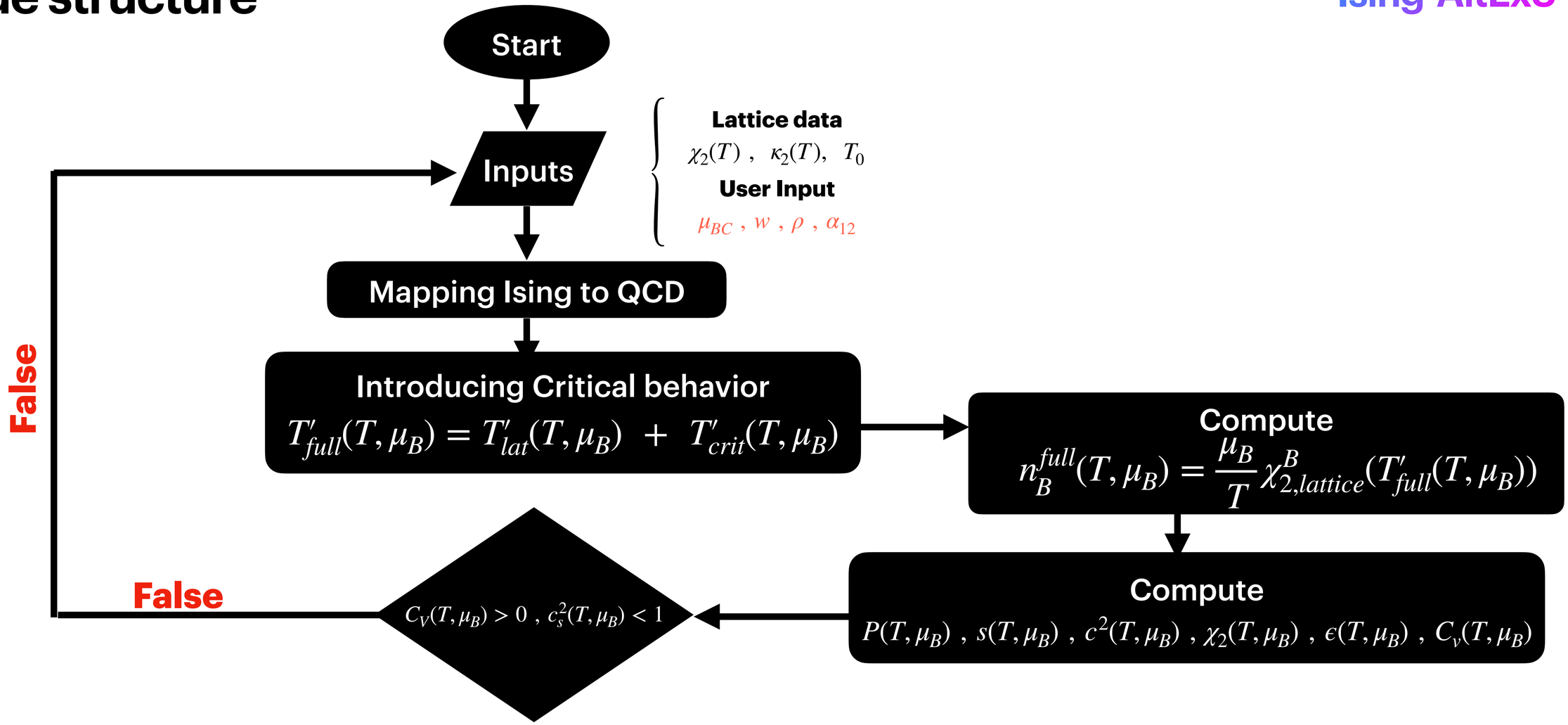


Code structure

Ising-AltExS



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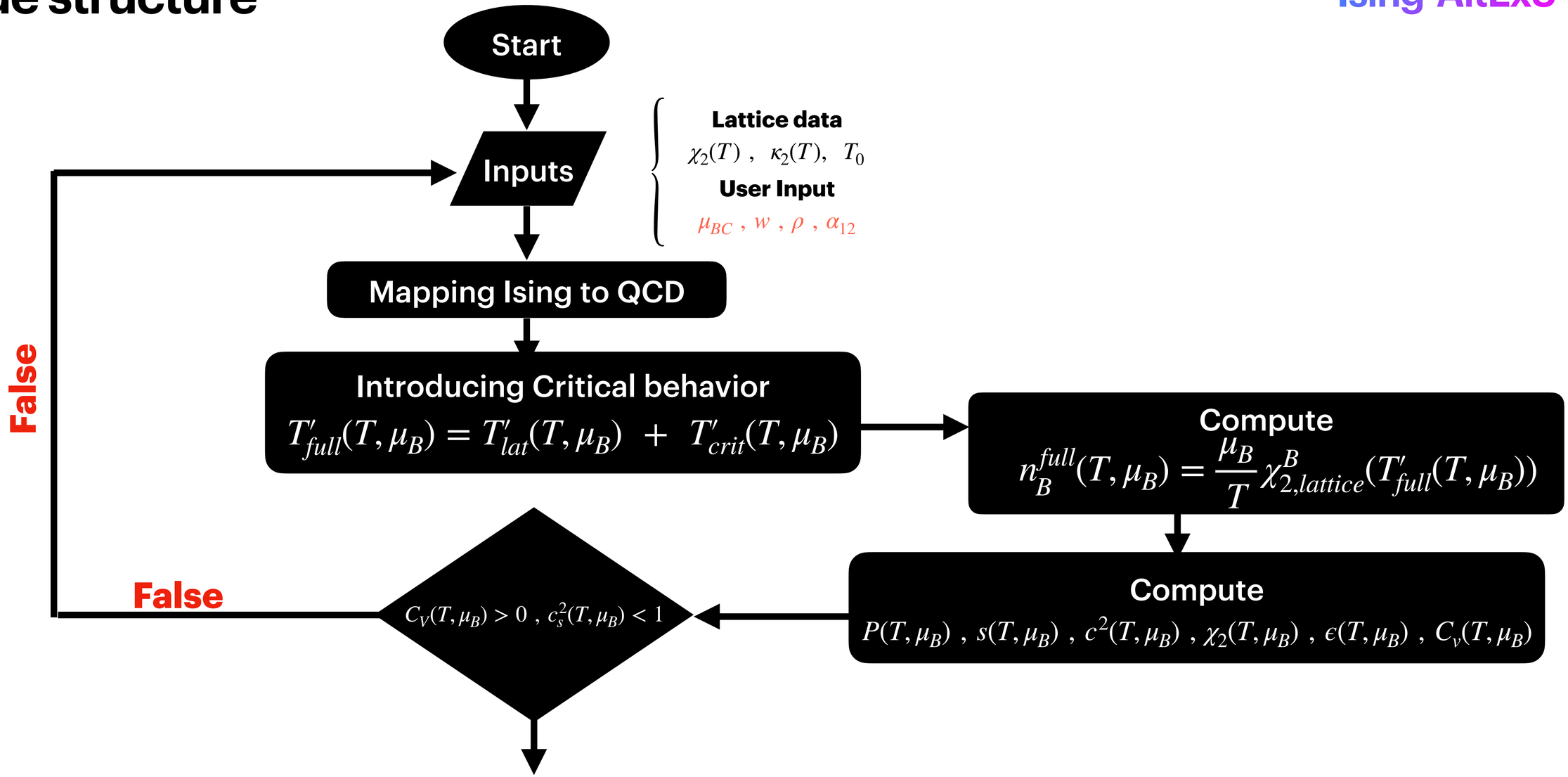

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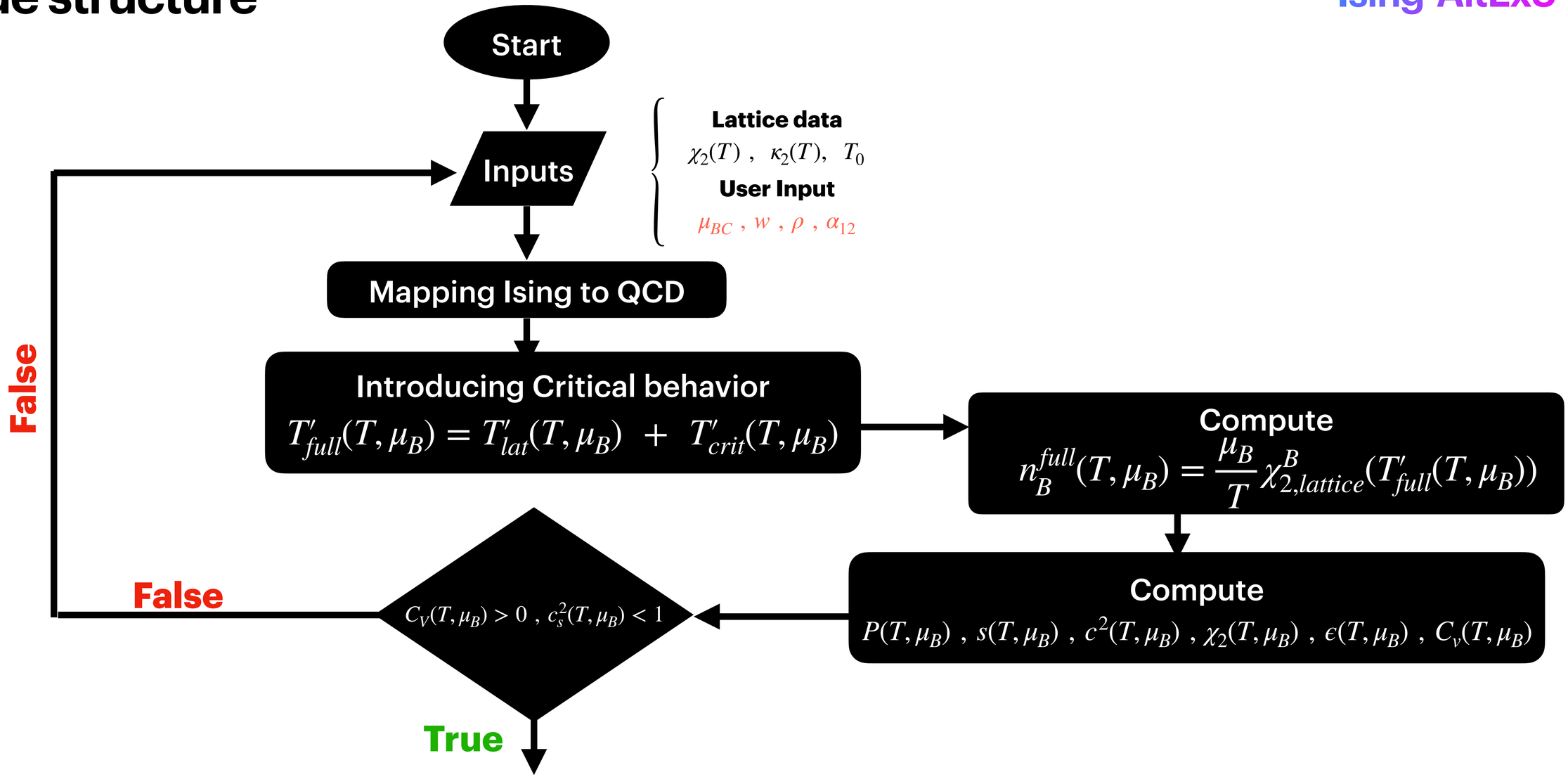

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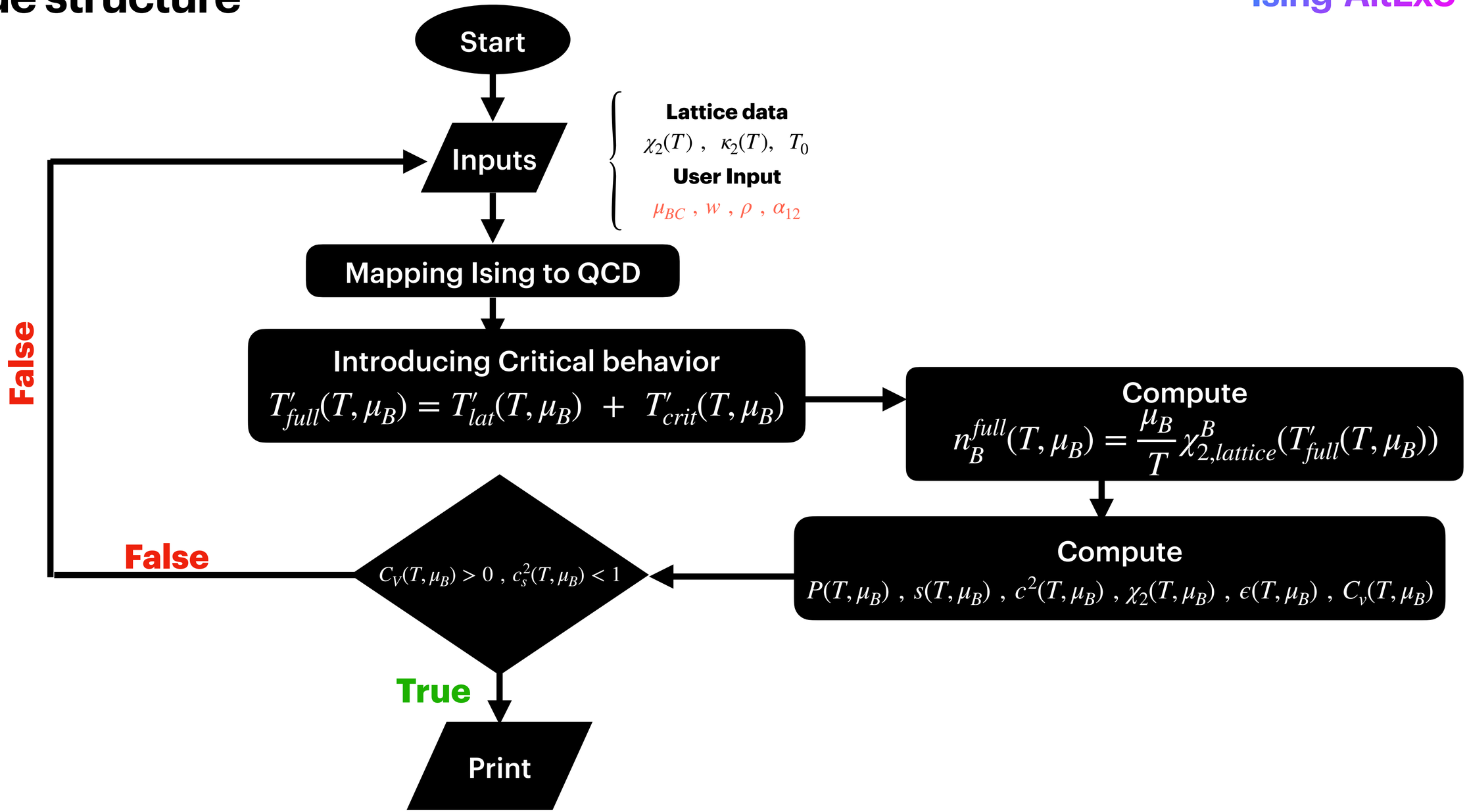

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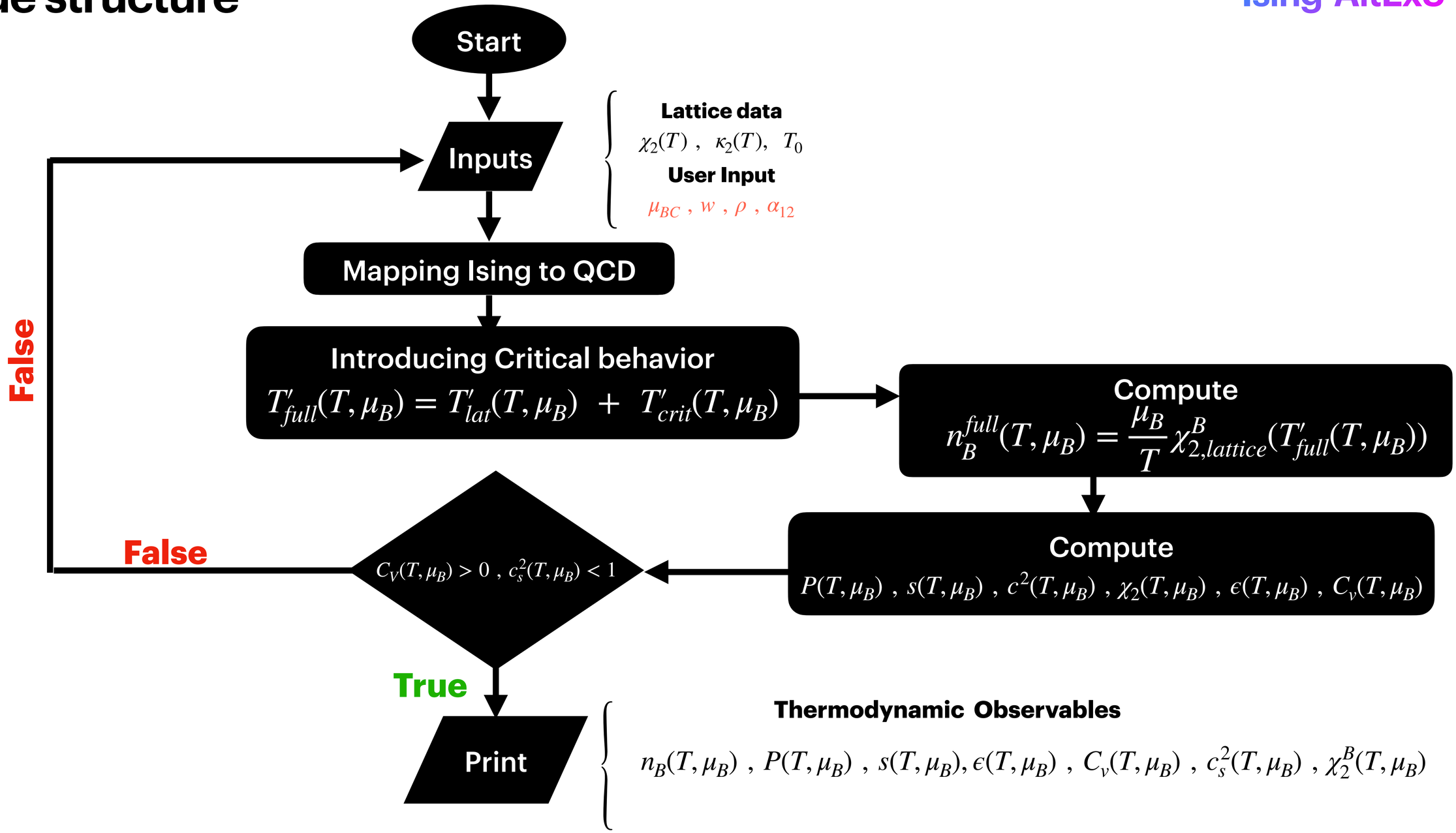
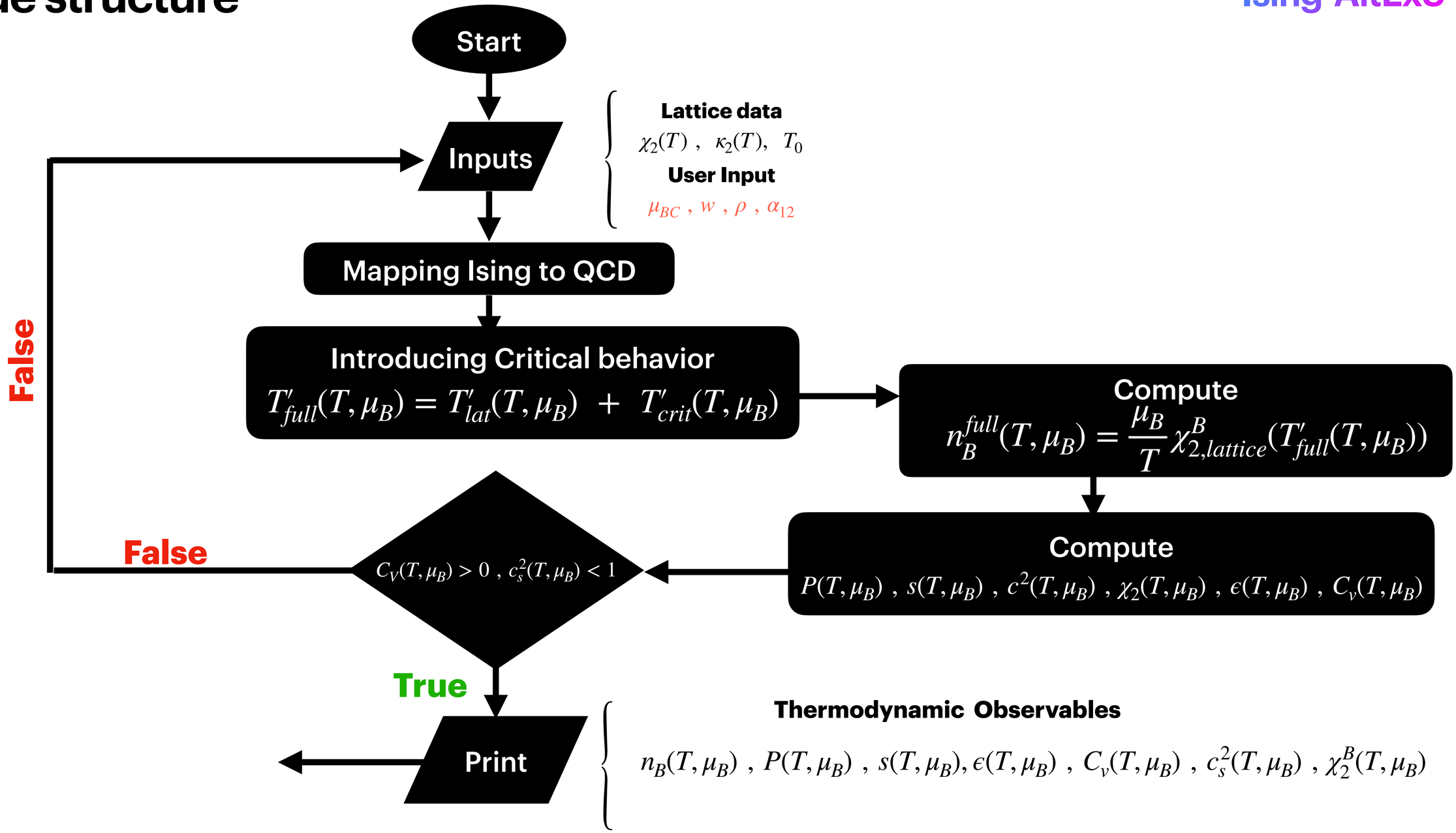
Ising-AltExS



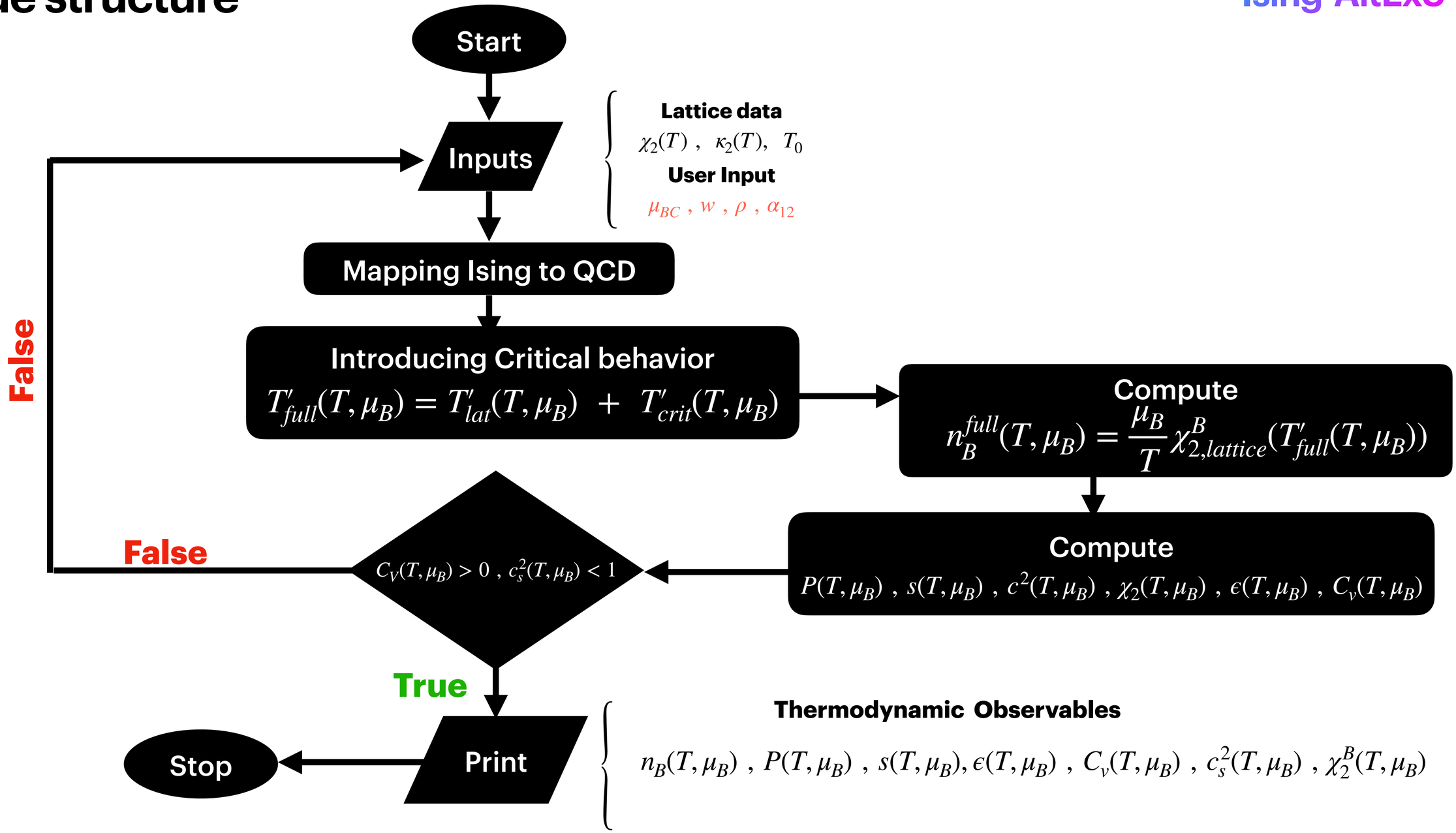
Code structure

Ising-AltExS



Code structure

Ising-AltExS



Code Status

Completed Tasks

Mathematica ✓ **(With few physics still under construction)**

C++ ~ **(Mapping and computes Baryon density)**

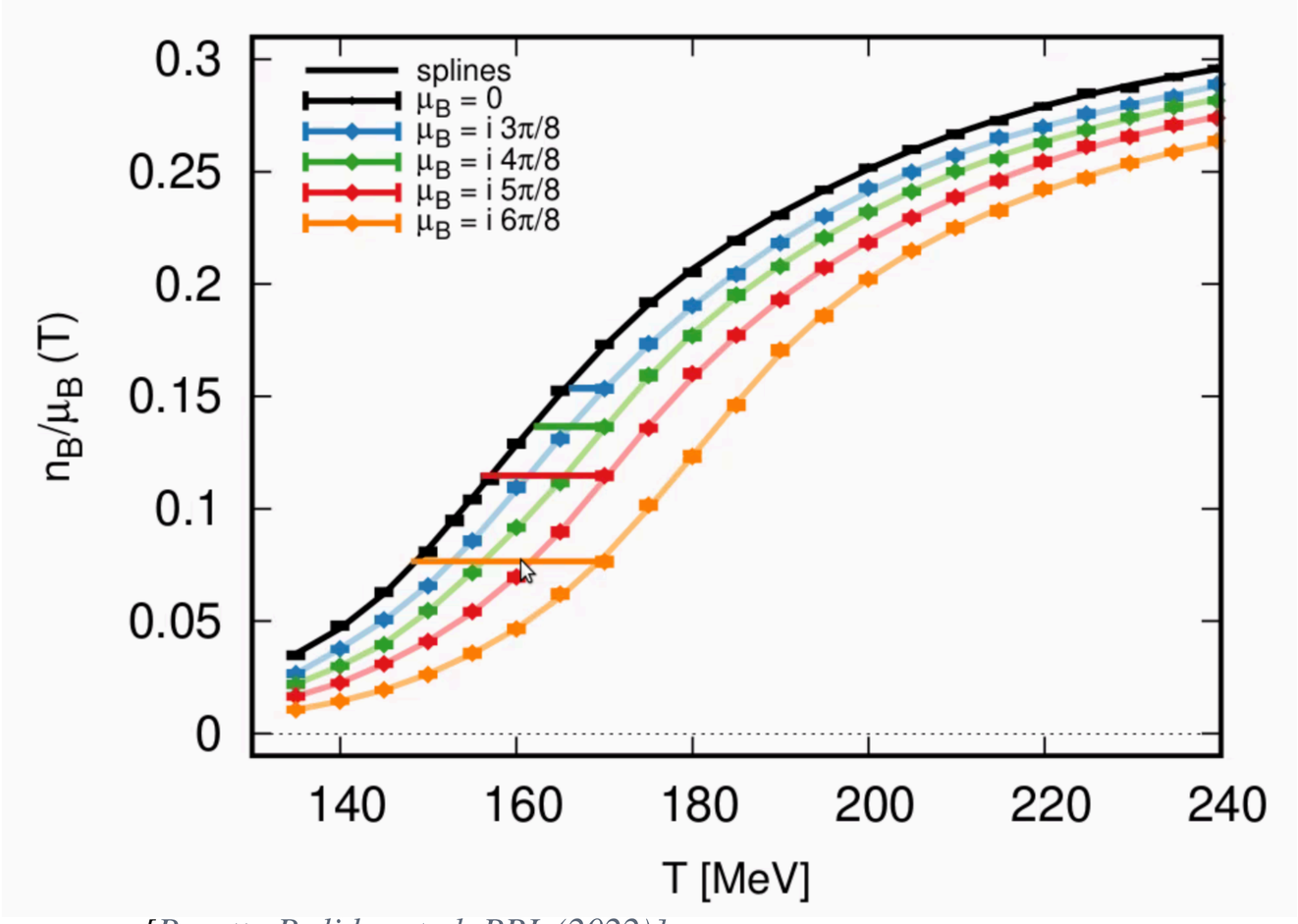
Remaining Tasks

- **Finishing the C++ code that computes all thermodynamics**
- **Defining the YAML specifications & manifest**
- **Create a Docker container**

Thank you !

Back up !

Shifting $T\dfrac{n_B(T,\mu_B)}{\mu_B}$ **w/ constant** $\kappa_2\left(\dfrac{\mu_B}{T}\right)^2$



[Parotto P slides et al. PRL (2022)]

Alternative Implementation

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi^B_{2,Lattice}(T'_{full}(T,\mu_B),0)$$

$$T'_{full}(T,\mu_B) = \underbrace{T'_{Lattice}(T,\mu_B)}_{\text{lowest order in } \left(\frac{\mu_B}{T}\right)} + \underbrace{T'_{crit}(T,\mu_B)}_{\text{higher orders in } \left(\frac{\mu_B}{T}\right)}$$

$$T'_{crit}(T,\mu_B) \approx T_0 + \left(\left.\frac{\partial\chi^B_{2,lattice}(T,0)}{\partial T}\right|_{T=T_0}\right)^{-1} \frac{n_B^{crit}(T,\mu_B)}{T^3(\mu_B/T)} f(T,\mu_B) + \ldots$$

$$f(T,\mu_B) = \frac{\mu_B^4}{\mu_{BC}^4}\frac{T^2}{T_0^2}$$

$$T'_{Lattice}(T,\mu_B) = T\left[1 + \left(\frac{\mu_B}{T}\right)^2 \kappa_2^{BB}(T) + \mathcal{O}\left(\frac{\mu_B}{T}\right)^4\right]$$